

Weight parameterization of simple modules for p -solvable groups

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1. Introduction

1.1. The *weights* for a finite group G with respect to a prime number p were introduced by Jon Alperin in [1] in order to formulate his celebrated conjecture. Explicitly, a *weight* of G is a pair (R, Y) formed by a p -subgroup R of G and by an isomorphism class Y of simple $kN_G(R)$ -modules with vertex R ; then, Alperin's Conjecture affirms that the number of G -conjugacy classes of weights of G coincides with the number of isomorphism classes of simple kG -modules, where k is an algebraically closed field of characteristic p . More precisely, Alperin's Conjecture involves the blocks of G as we explain below.

1.2. In the case that G is p -solvable, thirty years ago Tetsuro Okuyama [8] already proved that, for any p -subgroup R of G , the number of isomorphism classes Y of simple $kN_G(R)$ -modules with vertex R coincides with the number of isomorphism classes of simple kG -modules of vertex R , which clearly shows Alperin's Conjecture restricted to p -solvable groups. Once again, Okuyama's result actually involves the blocks of G . Note that, setting $\bar{N}_G(R) = N_G(R)/R$, a simple $kN_G(R)$ -module of vertex R is just the restriction of a *simple projective $k\bar{N}_G(R)$ -module*.

1.3. On the other hand, in [11, 6.4] we introduce a *multiplicity module* for any indecomposable kG -module M , and in [11, Lemma 9.9] we prove that M is determined by the triple formed by a *vertex* R , an *R -source* E and a *multiplicity module* V of M — an indecomposable projective $k_*\hat{\hat{N}}_G(R)_E$ -module where $\bar{N}_G(R)_E$ is the stabilizer of the isomorphism class of E in $\bar{N}_G(R)$, $\hat{\hat{N}}_G(R)_E$ is the *central k^* -extension* of $\bar{N}_G(R)_E$ determined by the action on $\text{End}_k(E)$, and $k_*\hat{\hat{N}}_G(R)_E$ is the corresponding *twisted* group algebra (cf. 2.5 below) — and that this correspondence actually defines a bijection between the set of isomorphism classes of indecomposable kG -modules and the set of G -conjugacy classes of triples (R, E, V) formed by a p -subgroup R of G , an indecomposable kR -module E of vertex R and an indecomposable projective $k_*\hat{\hat{N}}_G(R)_E$ -module V .

1.4. Moreover, if M is a simple kG -module then it follows from [9, Proposition 1.6] that V is actually a *simple projective $k_*\hat{\hat{N}}_G(R)_E$ -module*. But,

in the case that G is p -solvable and M is *primitive* — namely, not induced from any proper subgroup — it is well-known [17, Lemma 30.4] that there is a G -stable finite p' -subgroup K of $\text{End}_k(M)^*$ generating the k -algebra $\text{End}_k(M)$. Consequently, in this case $\text{End}_k(M)$ is actually a *Dade R -algebra* [13, 1.3]; in particular, $\bar{N}_G(R)$ -stabilizes the isomorphism class of E [13, 1.8] and it follows from [15, Theorem 9.21] that the central k^* -extension $k_*\hat{\bar{N}}_G(R)$ above is *split* — we are more explicit from 2.13 to 2.17 below.

1.5. That is to say, if G is p -solvable and M a primitive simple kG -module, then the pair formed by a vertex R and by the isomorphism class of the restriction to $N_G(R)$ of a multiplicity $k\bar{N}_G(R)$ -module V — after a choice of a splitting for the corresponding central k^* -extension — is actually a *weight* of G . More generally, since any simple kG -module is certainly induced from a primitive simple kH -module for some subgroup H of G , if G is p -solvable then $\text{End}_k(E)$ is always a *Dade R -algebra* for any vertex R and any R -source E of M ; hence, in this case, the central k^* -extension $k_*\hat{\bar{N}}_G(R)_E$ is always split and the corresponding multiplicity module V becomes a *simple projective $k\bar{N}_G(R)_E$ -module*.

1.6. In this paper, for a systematic choice of those splittings *via a polarization* [15, 9.5], on the one hand we exhibit a *natural bijection* — namely compatible with the action of the group of *outer automorphisms* of G — between the sets of isomorphism classes of simple kG -modules M and of G -conjugacy classes of weights (R, Y) of G . On the other hand, we determine the relationship between a multiplicity $k_*\hat{\bar{N}}_G(R)_E$ -module V and a simple $kN_G(R)$ -module U with vertex R in the class Y of the corresponding *weight* of G ; explicitly, there is a subgroup N of $N_G(R)_E$ containing R , a simple kN -module W of vertex R and, setting $\bar{N} = N/R$, a group homomorphism $\theta: \bar{N} \rightarrow k^*$ in such a way that, denoting by \bar{W} the corresponding $k\bar{N}$ -module and setting $\bar{W}_\theta = k_\theta \otimes_k \bar{W}$, we have

$$U \cong \text{Ind}_N^{N_G(R)}(W) \quad \text{and} \quad V \cong \text{Ind}_{\bar{N}}^{\bar{N}_G(R)_E}(\bar{W}_\theta) \quad 1.6.1.$$

The tools to carry out our purpose are mainly the *Fong reduction theorems* developed in [16]; as in that paper, it is handy — but *not* more general! — to work systematically with *k^* -groups with finite k^* -quotient G* [11, §5] — namely, with central k^* -extensions of G .

1.7. In 1994, when talking about this work at Beijing University, Zhang Jiping pointed out to us that Gabriel Navarro [7] already had given a bijection between the above sets of isomorphism classes of simple kG -modules and of G -conjugacy classes of *weights* for finite groups of *odd* order, and therefore *solvable*. In our Appendix we show that Navarro's bijection corresponds indeed to the bijection obtained for a particular choice of the *splittings* above, a choice which is only possible for groups of odd order.

2. Notations and quoted results

2.1. We fix a prime number p and an algebraically closed field k of characteristic p . We call k^* -group a group X endowed with an injective group homomorphism $\theta: k^* \rightarrow Z(X)$ [11, §5], and call k^* -quotient of (X, θ) the group $X/\theta(k^*)$; we denote by X° the k^* -group formed by X and by the composition of θ with the automorphism $k^* \cong k^*$ mapping $\lambda \in k^*$ on λ^{-1} ; we say that a k^* -group is *finite* whenever its k^* -quotient is finite. Usually, we denote by \hat{G} a k^* -group and by G its k^* -quotient, and we write $\lambda \cdot \hat{x}$ for the product of $\hat{x} \in \hat{G}$ and the image of $\lambda \in k^*$ in \hat{G} .

2.2. If \hat{G}' is a second k^* -group, we denote by $\hat{G} \hat{\times} \hat{G}'$ the quotient of the direct product $\hat{G} \times \hat{G}'$ by the image in $\hat{G} \times \hat{G}'$ of the *inverse* diagonal of $k^* \times k^*$, which has an obvious structure of k^* -group with k^* -quotient $G \times G'$; moreover, if $G = G'$ then we denote by $\hat{G} \star \hat{G}'$ the k^* -group obtained from the inverse image of $\Delta(G) \subset G \times G$ in $\hat{G} \hat{\times} \hat{G}'$, which is nothing but the so-called *sum* of both central k^* -extensions of G ; in particular, we have a *canonical* k^* -group isomorphism

$$\hat{G} \star \hat{G}^\circ \cong k^* \times G \quad 2.2.1.$$

A k^* -group homomorphism $\varphi: \hat{G} \rightarrow \hat{G}'$ is a group homomorphism which preserves the k^* -multiplication; moreover, if \hat{G} and \hat{G}' are isomorphic then the group $\text{Hom}(G, k^*)$ acts *regularly* over the set of isomorphisms $\psi: \hat{G} \cong \hat{G}'$ and we denote by ψ^θ the k^* -group isomorphism determined by $\theta \in \text{Hom}(G, k^*)$ and ψ . We denote by $k^*\text{-}\mathfrak{Gr}$ the category of k^* -groups.

2.3. Note that for any k -algebra A of finite dimension — just called *k-algebra* in the sequel — the group A^* of invertible elements has a canonical k^* -group structure; we call *point* of A any A^* -conjugacy class α of primitive idempotents of A and denote by $A(\alpha)$ the simple quotient of A determined by α , and by $\mathcal{P}(A)$ the set of *points* of A . If S is a simple algebra then $\text{Aut}_k(S)$ coincides with the k^* -quotient of S^* ; in particular, any finite group G acting on S determines — by *pull-back* — a k^* -group \hat{G} of k^* -quotient G , together with a k^* -group homomorphism [11, 5.7]

$$\rho: \hat{G} \longrightarrow S^* \quad 2.3.1.$$

2.4. If \hat{G} is a finite k^* -group, we call \hat{G} -interior algebra any k -algebra A endowed with a k^* -group homomorphism

$$\rho: \hat{G} \longrightarrow A^* \quad 2.4.1$$

and, as usual, we write $\hat{x} \cdot a$ and $a \cdot \hat{x}$ instead of $\rho(\hat{x})a$ and $a\rho(\hat{x})$ for any $\hat{x} \in \hat{G}$ and any $a \in A$; we say that A is *primitive* whenever the unity element is primitive in A^G . A \hat{G} -interior algebra homomorphism from A to another \hat{G} -interior algebra A' is a *not necessarily unitary* algebra homomorphism

$f: A \rightarrow A'$ fulfilling $f(\hat{x} \cdot a) = \hat{x} \cdot f(a)$ and $f(a \cdot \hat{x}) = f(a) \cdot \hat{x}$; we say that f is an *embedding* whenever $\text{Ker}(f) = \{0\}$ and $\text{Im}(f) = f(1)A'f(1)$. Occasionally, it is handy to consider the $(A'^{\hat{G}})^*$ -conjugacy class of f that we denote by \tilde{f} and call *exterior homomorphism* from A to A' ; note that the *exterior homomorphisms* can be composed [9, Definition 3.1]. For a k^* -group homomorphism $\varphi: \hat{G}' \rightarrow \hat{G}$, we denote by $\text{Res}_\varphi(A)$ the \hat{G}' -interior algebra defined by $\rho \circ \varphi$. Note that the conjugation induces an action of the k^* -quotient G of \hat{G} on A , so that A becomes an ordinary G -algebra; thus, all the *pointed group* language developed in [9] applies to \hat{G} -interior algebras.

2.5. Namely, for any k^* -subgroup \hat{H} of \hat{G} , a *point* α of \hat{H} on A is just a point of the k -algebra A^H , and the pair \hat{H}_α is a *pointed k^* -group* on A ; we denote by $A(\hat{H}_\alpha)$ the simple quotient $A^H(\alpha)$ and, setting

$$\bar{N}_G(\hat{H}_\alpha) = N_G(\hat{H}_\alpha)/H \quad \text{and} \quad A(\hat{H}_\alpha) = \text{End}_k(V_\alpha) \quad 2.5.1,$$

by $\hat{N}_G(\hat{H}_\alpha)$ the k^* -group determined by the action of $\bar{N}_G(\hat{H}_\alpha)$ on $A(\hat{H}_\alpha)$, so that V_α becomes a $\hat{N}_G(\hat{H}_\alpha)$ -module called the *multiplicity $\hat{N}_G(\hat{H}_\alpha)$ -module* of \hat{H}_α [11, 6.4]. For any $i \in \alpha$, iAi has an evident structure of \hat{H} -interior algebra mapping $\hat{x} \in \hat{H}$ on $\hat{x} \cdot i = i \cdot \hat{x}$ and we denote by A_α one of these mutually $(A^H)^*$ -conjugate \hat{H} -interior algebras. If A' is another \hat{G} -interior algebra and $f: A \rightarrow A'$ a \hat{G} -interior algebra embedding, $f(\alpha)$ is contained in a unique point α' of \hat{H} on A' , usually identified with α , and f induces a k -algebra embedding, a k^* -group isomorphism and an \hat{H} -interior algebra isomorphism

$$A(\hat{H}_\alpha) \longrightarrow A'(\hat{H}_{\alpha'}) \quad , \quad \hat{N}_G(\hat{H}_\alpha) \cong \hat{N}_G(\hat{H}_{\alpha'}) \quad \text{and} \quad A_\alpha \xrightarrow{f_\alpha^{\alpha'}} A_{\alpha'} \quad 2.5.2.$$

2.6. A second pointed k^* -group \hat{K}_β on A is *contained* in \hat{H}_α if \hat{K} is a k^* -subgroup of \hat{H} and, for any $i \in \alpha$, there is $j \in \beta$ such that $ij = j = ji$; then, it is quite clear that the $(A^K)^*$ -conjugation induces a \hat{K} -interior algebra embedding

$$f_\beta^\alpha: A_\beta \longrightarrow \text{Res}_{\hat{K}}^{\hat{H}}(A_\alpha) \quad 2.6.1.$$

More generally, we say that an injective k^* -group homomorphism $\varphi: \hat{K} \rightarrow \hat{H}$ is an *A-fusion from \hat{K}_β to \hat{H}_α* whenever there is a \hat{K} -interior algebra embedding

$$f_\varphi: A_\beta \longrightarrow \text{Res}_\varphi(A_\alpha) \quad 2.6.2$$

such that the inclusion $A_\beta \subset A$ and the composition of f_φ with the inclusion $A_\alpha \subset A$ are A^* -conjugate; then, the *exterior embedding* \tilde{f}_φ is uniquely determined [10, 2.8]. We denote by $F_A(\hat{K}_\beta, \hat{H}_\alpha)$ the set of \hat{H} -conjugacy classes of

A -fusions from \hat{K}_β to \hat{H}_α [12, Definition 2.5] and we simply set

$$F_A(\hat{H}_\alpha) = F_A(\hat{H}_\alpha, \hat{H}_\alpha) \quad 2.6.3;$$

note that the conjugation in \hat{G} induces a canonical group homomorphism

$$\bar{N}_G(\hat{H}_\alpha) \longrightarrow F_A(\hat{H}_\alpha) \quad 2.6.4.$$

If A' is another \hat{G} -interior algebra and $f: A \rightarrow A'$ a \hat{G} -interior algebra embedding, it follows from [10, Proposition 2.14] that we have

$$F_A(\hat{K}_\beta, \hat{H}_\alpha) = F_{A'}(\hat{K}_\beta, \hat{H}_\alpha) \quad 2.6.5.$$

2.7. Note that any p -subgroup P of \hat{G} can be identified with its image in G and determines the k^* -subgroup $k^* \cdot P \cong k^* \times P$ of \hat{G} ; as usual, we consider the *Brauer quotient* and the *Brauer algebra homomorphism*

$$\text{Br}_P : A^P \longrightarrow A(P) = A^P / \sum_Q A_Q^P \quad 2.7.1,$$

where Q runs over the set of proper subgroups of P , and call *local* any point γ of P on A not contained in $\text{Ker}(\text{Br}_P)$; recall that all the *maximal local pointed groups* P_γ on A contained in \hat{H}_α — called *defect pointed groups* of \hat{H}_α — are mutually H -conjugate [9, Theorem 1.2], and that the k -algebras A_α and A_γ are *Morita equivalent* [9, Corollary 3.5]. If $A_\gamma = iAi$ for $i \in \gamma$, it follows from [10, Corollary 2.13] that we have a group homomorphism

$$F_A(P_\gamma) \longrightarrow N_{A_\gamma^*}(P \cdot i) / P \cdot (A_\gamma^P)^* \quad 2.7.2$$

and we consider the k^* -group $\hat{F}_A(P_\gamma)$ defined by the *pull-back*

$$\begin{array}{ccc} F_A(P_\gamma) & \longrightarrow & N_{A_\gamma^*}(P \cdot i) / P \cdot (A_\gamma^P)^* \\ \uparrow & & \uparrow \\ \hat{F}_A(P_\gamma) & \longrightarrow & N_{A_\gamma^*}(P \cdot i) / P \cdot (i + J(A_\gamma^P)) \end{array} \quad 2.7.3.$$

2.8. Then, from [11, Proposition 6.12] suitably extended to k^* -groups, it follows that the group homomorphism 2.6.4 can be lifted to a canonical k^* -group homomorphism

$$\hat{N}_G(P_\gamma) * \bar{N}_{\hat{G}}(P_\gamma)^\circ \longrightarrow \hat{F}_A(P_\gamma)^\circ \quad 2.8.1$$

which, for any $\hat{x} \in \bar{N}_{\hat{G}}(P_\gamma) = N_{\hat{G}}(P_\gamma)/P$ and any $a \in (A^P)^*$ having the same action on $A(P_\gamma)$, maps the element $(x, \bar{a}) * \hat{x}^{-1}$ of $\hat{N}_G(P_\gamma) * \bar{N}_{\hat{G}}(P_\gamma)^\circ$ on the pair [11, Proposition 6.10]

$$(\tilde{x}^{-1}, \overline{i(\hat{x}^{-1} \cdot a)i}) \in \hat{F}_A(P_\gamma) \quad 2.8.2,$$

where x denotes the image of \hat{x} in $\bar{N}_G(P_\gamma)$, \bar{a} the image of a in $A(P_\gamma)$, \tilde{x} the image of x in $F_A(P_\gamma)$ via homomorphism 2.6.4 and $\overline{i(\hat{x}^{-1} \cdot a)i}$ the image of $i(\hat{x}^{-1} \cdot a)i$ in the right-hand bottom of diagram 2.7.3.

2.9. If A' is another \hat{G} -interior algebra and $f : A \rightarrow A'$ a \hat{G} -interior algebra embedding, it follows from [11, Proposition 6.8] that, denoting by γ' the point of P on A' containing $f(\gamma)$, we have a canonical k^* -group isomorphism

$$\hat{F}_{\tilde{f}}(P_\gamma) : \hat{F}_A(P_\gamma) \cong \hat{F}_{A'}(P_{\gamma'}) \quad 2.9.1$$

which, according to [11, Proposition 6.21], is compatible with the corresponding k^* -group homomorphisms 2.8.1 and 2.5.2. More precisely, let Q_δ be another local pointed group on A and denote by δ' the point of Q on A' containing $f(\delta)$; if there is a group isomorphism $\varphi : Q \cong P$ which is an A -fusion from Q_δ to P_γ then, according to equality 2.6.5 above, φ is also an A' -fusion from $Q_{\delta'}$ to $P_{\gamma'}$, so that we have two Q -interior algebra isomorphisms

$$f_\varphi : A_\delta \cong \text{Res}_\varphi(A_\gamma) \quad \text{and} \quad f'_\varphi : A'_{\delta'} \cong \text{Res}_\varphi(A'_{\gamma'}) \quad 2.9.2$$

and the uniqueness of the *exterior isomorphisms* \tilde{f}_φ and \tilde{f}'_φ forces the equality

$$\tilde{f}'_\varphi \circ \tilde{f}_\delta^{\delta'} = \text{Res}_\varphi(\tilde{f}_{\gamma'}^{\gamma'}) \circ \tilde{f}_\varphi \quad 2.9.3.$$

In particular, since by the very definition we have

$$\hat{F}_{\text{Res}_\varphi(A_\gamma)}(Q_\delta) = \hat{F}_A(P_\gamma) \quad \text{and} \quad \hat{F}_{\text{Res}_\varphi(A'_{\gamma'})}(Q_{\delta'}) = \hat{F}_{A'}(P_{\gamma'}) \quad 2.9.4,$$

we get the following commutative diagram of k^* -group isomorphisms

$$\begin{array}{ccc} \hat{F}_A(Q_\delta) & \xrightarrow{\hat{F}_{\tilde{f}_\varphi}(Q_\delta)} & \hat{F}_A(P_\gamma) \\ \hat{F}_{\tilde{f}}(Q_\delta) \wr & & \wr \hat{F}_{\tilde{f}}(P_\gamma) \\ \hat{F}_{A'}(Q_{\delta'}) & \xrightarrow{\hat{F}_{\tilde{f}'_\varphi}(Q_{\delta'})} & \hat{F}_{A'}(P_{\gamma'}) \end{array} \quad 2.9.5.$$

2.10. It is clear that the inclusion $k^* \subset k$ determines a k -algebra homomorphism to k from the group algebra kk^* of the group k^* , so that k becomes a kk^* -algebra; for any finite k^* -group \hat{G} , it is clear that the group algebra $k\hat{G}$ of the group \hat{G} is also a kk^* -algebra and then, we call k^* -group algebra of \hat{G} the algebra

$$k_*\hat{G} = k \otimes_{kk^*} k\hat{G} \quad 2.10.1;$$

note that the dimension of $k_*\hat{G}$ is equal to $|\hat{G}|$. Coherently, a *block* of \hat{G} is a primitive idempotent b of the center $Z(k_*\hat{G})$, so that $\alpha = \{b\}$ is a point of \hat{G} on $k_*\hat{G}$; as usual, we denote by $\text{Irr}_k(\hat{G}, b)$ the set of *Brauer characters* of all the simple $k_*\hat{G}b$ -modules, which corresponds bijectively with the set of points $\mathcal{P}(k_*\hat{G}b)$.

2.11. Recall that for any p -subgroup P of \hat{G} we have [11, 2.10.2 and Proposition 5.15]

$$(k_*\hat{G})(P) \cong k_*C_{\hat{G}}(P) \quad 2.11.1;$$

in particular, if P is normal in G , since the kernel of the obvious k -algebra homomorphism $k_*\hat{G} \rightarrow k_*(\hat{G}/P)$ is contained in the *radical* $J(k_*\hat{G})$ and contains $\text{Ker}(\text{Br}_P)$, this isomorphism implies that *any point of P on $k_*\hat{G}$ is local*. Moreover, it follows from [10, Theorem 3.1] that we have

2.11.2 For any pair of local pointed groups P_γ and Q_δ on $k_*\hat{G}$, a $k_*\hat{G}$ -fusion from Q_δ to P_γ coincides with the conjugation by an element $x \in G$ such that $Q_\delta \subset (P_\gamma)^x$.

2.12. If \hat{G} is a finite k^* -group, A a \hat{G} -interior algebra and \hat{H} a k^* -subgroup of \hat{G} , as usual we denote by $\text{Res}_{\hat{H}}^{\hat{G}}(A)$ the corresponding \hat{H} -interior algebra. Conversely, for any \hat{H} -interior algebra B , we consider the *induced G -interior algebra*

$$\text{Ind}_{\hat{H}}^{\hat{G}}(B) = k_*\hat{G} \otimes_{k_*\hat{H}} B \otimes_{k_*\hat{H}} k_*\hat{G} \quad 2.12.1,$$

where the distributive product is defined by the *formula*

$$(\hat{x} \otimes b \otimes \hat{y})(\hat{x}' \otimes b' \otimes \hat{y}') = \begin{cases} \hat{x} \otimes b \cdot \hat{y} \hat{x}' \cdot b' \otimes \hat{y}' & \text{if } \hat{y} \hat{x}' \in \hat{H} \\ 0 & \text{otherwise} \end{cases} \quad 2.12.2$$

for any $\hat{x}, \hat{y}, \hat{x}', \hat{y}' \in \hat{G}$ and any $b, b' \in B$, and where we map $\hat{x} \in \hat{G}$ on the element

$$\sum_{\hat{y}} \hat{x} \hat{y} \otimes 1_B \otimes \hat{y}^{-1} = \sum_{\hat{y}} \hat{y} \otimes 1_B \otimes \hat{y}^{-1} \hat{x} \quad 2.12.3,$$

$\hat{y} \in \hat{G}$ running over a set of representatives for \hat{G}/\hat{H} .

2.13. For a finite p -group P , we call *Dade P -algebra* [13, 1.3] a simple algebra S endowed with an action of P which stabilizes a basis of S containing the unity element; actually, the action of P on S can be lifted to a unique group homomorphism $P \rightarrow S^*$ and usually we consider S as a P -interior algebra; moreover, the *Brauer quotient* $S(P)$ is also a simple k -algebra [13, 1.8] which implies that P has a unique local point ρ on S that very often we omit, respectively writing $F_S(P)$ and $\hat{F}_S(P)$ instead of $F_S(P_\rho)$ and $\hat{F}_S(P_\rho)$. Recall that two Dade P -algebras S and S' are *similar* if S can be *embedded* (cf. 2.4) in the *tensor product* $\text{End}(N) \otimes_k S'$ for a suitable kP -module N with a P -stable basis [13, 1.5 and 2.5.1]; we denote by $\mathcal{D}_k(P)$ the set of *similarity classes* and the *tensor product* induces a *group structure* on $\mathcal{D}_k(P)$ — called the *Dade group* of P — where the opposite P -algebra S° determines the inverse of the *similarity* class of S .

2.14. As in [15, 9.3], it is handy to consider the category \mathfrak{D}_k where the objects are the pairs (P, S) formed by a finite p -group P and by a Dade P -algebra S , and where a morphism from (P, S) to a second \mathfrak{D}_k -object

(P', S') are the pairs (π, f) formed by a surjective group homomorphism $\pi : P \rightarrow P'$ such that $\text{Ker}(\pi)$ is $F_S(P)$ -stable, and by a P -interior algebra embedding

$$f : \text{Res}_\pi(S') \longrightarrow S \quad 2.14.1.$$

Then, we have functors \mathfrak{f} and $\hat{\mathfrak{f}}$ mapping (P, S) on $F_S(P)$ and $\hat{F}_S(P)$ [15, 9.5], together a *natural map* $\hat{\mathfrak{f}} \rightarrow \mathfrak{f}$ mapping (P, S) on the structural homomorphism

$$\hat{F}_S(P) \longrightarrow F_S(P) \quad 2.14.2.$$

2.15. As in [15, 9.5], we call *polarization* any *natural map* ω from the functor $\hat{\mathfrak{f}} : \mathfrak{D}_k \rightarrow k^*\text{-}\mathfrak{Gr}$ above to the *trivial* one — namely, to the functor mapping (P, S) on k^* and (π, f) on id_{k^*} — such that if T is a P -algebra with trivial P -action then ω maps (P, T) on the first projection in the isomorphism

$$\hat{F}_T(P) \cong k^* \times F_T(P) \quad 2.15.1$$

obtained from the corresponding pull-back 2.7.3. The point is that, according to [15, Theorem 9.21], there exists such a *natural map*, and we will construct a bijection as announced above from any choice of a *polarization* ω , namely from any choice, in a *coherent* way, of a k^* -group homomorphism

$$\omega_{(P, S)} : \hat{F}_S(P) \longrightarrow k^* \quad 2.15.2$$

for any \mathfrak{D}_k -object (P, S) . A first application of this existence concerns the *multiplicity modules* of the indecomposable $k_*\hat{G}$ -modules M having a vertex P and a P -source N such that $\text{End}_k(N)$ is a Dade P -algebra.

Lemma 2.16. *Let \hat{G} be a finite k^* -group, M an indecomposable $k_*\hat{G}$ -module, P a vertex and N a P -source of M ; let us denote by P_N the local pointed group on the \hat{G} -interior algebra $\text{End}_k(M)$ determined by the pair (P, N) . If $\text{End}_k(N)$ is a Dade P -algebra then the action of $\bar{N}_G(P_N)$ on the simple quotient $(\text{End}_k(M))(P_N)$ can be lifted to a k^* -group homomorphism*

$$\bar{N}_{\hat{G}}(P_N) \longrightarrow (\text{End}_k(M))(P_N)^* \quad 2.16.1.$$

Proof: In any case, this action determines a k^* -group $\hat{N}_G(P_N)$ and we have a canonical k^* -group homomorphism (cf. 2.8.1)

$$\hat{N}_G(P_N) * \bar{N}_{\hat{G}}(P_N)^\circ \longrightarrow \hat{F}_{\text{End}_k(N)}(P_N)^\circ \quad 2.16.2;$$

but, if $\text{End}_k(N)$ is a Dade P -algebra, the existence of a *polarization* implies that, in particular, we have

$$\hat{F}_{\text{End}_k(N)}(P) \cong k^* \times F_{\text{End}_k(N)}(P) \quad 2.16.3;$$

consequently, we get $\hat{N}_G(P_N) \cong \bar{N}_{\hat{G}}(P_N)$. We are done.

2.17. More generally, if S is a Dade P -algebra and A a P -interior algebra, it follows from [12, Theorem 5.3] that, for any subgroup Q of P , we have a canonical bijection between the sets of local points of Q on A and on $S \otimes_k A$; moreover, if A admits a $P \times P$ -stable basis by the multiplication on both sides, where $P \times \{1\}$ and $\{1\} \times P$ act freely, it follows from [6, Lemma 1.17] that, for any pair of local pointed groups Q_δ and R_ε on A , we have

$$F_{S \otimes_k A}(R_{S \times \varepsilon}, Q_{S \times \delta}) = F_S(R, Q) \cap F_A(R_\varepsilon, Q_\delta) \quad 2.17.1$$

where $S \times \varepsilon$ and $S \times \delta$ denote the corresponding local points of R and Q on $S \otimes_k A$; in this case, since the choice of a *polarization* ω determines a k^* -group homomorphism

$$\omega_{(Q, \text{Res}_Q^P(S))} : \hat{F}_S(Q) \longrightarrow k^* \quad 2.17.2,$$

it follows from [12, Proposition 5.11] that the inclusion of $F_{S \otimes_k A}(Q_{S \times \delta})$ in $F_A(Q_\delta)$ can be lifted to a k^* -group homomorphism determined by ω

$$\Phi_S^\omega(Q_\delta) : \hat{F}_{S \otimes_k A}(Q_{S \times \delta}) \longrightarrow \hat{F}_A(Q_\delta) \quad 2.17.3.$$

More precisely, as in 2.9 above, if A' is a P -interior algebra and $f : A \rightarrow A'$ a P -interior algebra embedding, denoting by δ' the point of Q on A' containing $f(\delta)$, from [12, Proposition 5.11] we still get the following commutative diagram of k^* -group homomorphism

$$\begin{array}{ccc} \hat{F}_{S \otimes_k A}(Q_{S \times \delta}) & \xrightarrow{\Phi_S^\omega(Q_\delta)} & \hat{F}_A(Q_\delta) \\ \hat{F}_{\text{id}_S \otimes f}(Q_{Q \times \delta}) \wr & & \wr \hat{F}_{\bar{f}}(Q_\delta) \\ \hat{F}_{S \otimes_k A'}(Q_{S \times \delta'}) & \xrightarrow{\Phi_{S'}^\omega(Q_{\delta'})} & \hat{F}_{A'}(Q_{\delta'}) \end{array} \quad 2.17.4.$$

3. The weights revisited

3.1. Let \hat{G} be a finite k^* -group; we say that a local pointed group Q_δ on $k_*\hat{G}$ is *selfcentralizing* if $C_P(Q) = Z(Q)$ for any local pointed group P_γ on $k_*\hat{G}$ containing Q_δ , and that it is a *radical* whenever it is selfcentralizing and we have

$$\mathbb{O}_p(F_{k_*\hat{G}}(Q_\delta)) = \{1\} \quad 3.1.1.$$

Recall that, according to [15, 4.8 and Corollary 7.3], Q_δ is *selfcentralizing* if, denoting by f the block of $C_{\hat{G}}(Q)$ determined by δ , the image \bar{f} of f in the k^* -group algebra of $\bar{C}_{\hat{G}}(Q) = C_{\hat{G}}(Q)/Z(Q)$ is a block of *defect zero*; note that, in this case, δ is the unique local point of Q on $k_*\hat{G}$ determining the block f .

3.2. As mentioned in 1.2 above, a *weight* (R, Y) of \hat{G} is formed by a p -subgroup R of \hat{G} and by the isomorphism class Y of the restriction to $N_{\hat{G}}(R)$ of a *simple projective* $k_*\bar{N}_{\hat{G}}(R)$ -module V , where we set $\bar{N}_{\hat{G}}(R) = N_{\hat{G}}(R)/R$;

let us denote by $\text{Wgt}_k(\hat{G})$ the set of G -conjugacy classes of *weights* of \hat{G} . Then, the restriction of V to $\bar{C}_{\hat{G}}(R) \triangleleft \bar{N}_{\hat{G}}(R)$ is a *semisimple projective* $k_*\bar{C}_{\hat{G}}(R)$ -module and thus any simple direct summand W of $\text{Res}_{\bar{C}_{\hat{G}}(R)}^{\bar{N}_{\hat{G}}(R)}(V)$ is also projective, so that it determines the unique *local point* ε of R on $k_*\hat{G}$ (cf. 2.11.1) in a block \bar{g} of defect zero of $\bar{C}_{\hat{G}}(R)$; that is to say, W determines a selfcentralizing pointed group R_ε on $k_*\hat{G}$ and the stabilizer of the isomorphism class of W in $N_{\hat{G}}(R)$ coincides with $N_{\hat{G}}(R_\varepsilon)$.

3.3. Moreover, it follows from isomorphism 2.11.1 that we have

$$(k_*\hat{G})(R_\varepsilon) \cong k_*\bar{C}_{\hat{G}}(R)\bar{g} \cong \text{End}_k(W) \quad 3.3.1$$

and from 2.5 we know that W becomes an $\hat{N}_G(R_\varepsilon)$ -module; then, since the $\bar{N}_{\hat{G}}(R)$ -interior algebra $\text{End}_k(V)$ is isomorphic to a suitable *block algebra* of $\bar{N}_{\hat{G}}(R)$, and since we have (cf. 2.10)

$$N_{\hat{G}}(R_\varepsilon)/C_{\hat{G}}(R) \cong F_{k_*\hat{G}}(R_\varepsilon) \quad 3.3.2,$$

it follows from [16, Theorem 3.7] and from 2.8 above that, for a suitable *simple projective* $k_*\hat{F}_{k_*\hat{G}}(R_\varepsilon)$ -module U restricted to $\hat{N}_G(R_\varepsilon)^\circ * \bar{N}_{\hat{G}}(R_\varepsilon)$ via homomorphism 2.8.1, we obtain

$$V \cong \text{Ind}_{\bar{N}_{\hat{G}}(R_\varepsilon)}^{\bar{N}_{\hat{G}}(R)}(W \otimes_k U) \quad 3.3.3;$$

in particular, we get

$$\mathbb{O}_p(F_{k_*\hat{G}}(R_\varepsilon)) = \{1\} \quad 3.3.4,$$

so that R_ε is a *radical* pointed group.

3.4. Conversely, if R_ε is a radical pointed group on $k_*\hat{G}$ and U a *simple projective* $k_*\hat{F}_{k_*\hat{G}}(R_\varepsilon)$ -module, it is easily checked that the restriction of U to $\hat{N}_G(R_\varepsilon)^\circ * \bar{N}_{\hat{G}}(R_\varepsilon)$ throughout homomorphism 2.8.1, together with a *multiplicity* $\hat{N}_G(R_\varepsilon)$ -module W of R_ε define a *simple projective* $k_*\bar{N}_{\hat{G}}(R)$ -module via the tensor product and the induction as in 3.3.3. In conclusion, we have proved that

3.4.1. *The above correspondence between the sets of G -conjugacy classes of weights (R, Y) of \hat{G} and of pairs (R_ε, X) formed by a radical pointed group R_ε on $k_*\hat{G}$ and by an isomorphism class X of simple projective $k_*\hat{F}_{k_*\hat{G}}(R_\varepsilon)$ -modules is bijective.*

Let us call *b-weight* of \hat{G} any pair (R_ε, X) formed by a radical pointed group R_ε on $k_*\hat{G}$ and by an isomorphism class X of simple projective

$k_*\hat{F}_{k_*\hat{G}}(R_\varepsilon)$ -modules, and let us denote by $\text{Wgt}_k(\hat{G}, b)$ the set of G -conjugacy classes of b -weights of \hat{G} ; thus, statement 3.4.1 affirms that we have a canonical bijection

$$\text{Wgt}_k(\hat{G}) \cong \bigsqcup_b \text{Wgt}_k(\hat{G}, b) \quad 3.4.2$$

where b runs over the set of blocks of \hat{G} ; in particular, any weight of \hat{G} determines a block.

4. Fitting pointed groups

4.1. Let us say that a finite k^* -group \hat{G} is p -solvable if the k^* -quotient G of \hat{G} is so; it is in this case that the following definition is actually useful. We call *Fitting pointed group* of \hat{G} any radical pointed group Q_δ on $k_*\hat{G}$ fulfilling the following condition

4.1.1. *For any local pointed groups P_γ and R_ε on $k_*\hat{G}$ such that P_γ contains Q_δ and R_ε , any $k_*\hat{G}$ -fusion from R_ε to P_γ coincides with the conjugation by an element $x \in N_G(Q_\delta)$ fulfilling $R_\varepsilon \subset (P_\gamma)^x$.*

Note that this condition implies that a *Fitting pointed group* Q_δ of \hat{G} is normal in any local pointed group P_γ containing Q_δ .

Proposition 4.2. *Let \hat{G} be a finite k^* -group and Q_δ a Fitting pointed group of \hat{G} . If a local pointed group P_γ on $k_*\hat{G}$ contains both Q_δ and a radical pointed group R_ε on $k_*\hat{G}$, then R_ε contains Q_δ . In particular, Q_δ is the unique Fitting pointed group of \hat{G} contained in P_γ .*

Proof: We already know that Q_δ is normal in P_γ and therefore the product $Q \cdot R$ is a subgroup of P ; but, any element $y \in N_G(R_\varepsilon)$ induces by conjugation a $k_*\hat{G}$ -fusion from R_ε to P_γ and therefore, according to condition 4.1.1, this $k_*\hat{G}$ -fusion is also induced by an element $x \in N_G(Q_\delta)$; in particular, the image of $N_Q(R_\varepsilon)$ in $F_{k_*\hat{G}}(R_\varepsilon)$ is a normal p -subgroup and therefore it is trivial.

On the other hand, it follows from [2, Theorem 1.8] and from 3.1 above that ε is the unique local point of R on $k_*\hat{G}$ such that $R_\varepsilon \subset P_\gamma$, and thus we have $N_Q(R_\varepsilon) = N_Q(R)$; moreover, since R_ε is selfcentralizing, we still have $C_P(R) = Z(R)$ and therefore $\bar{N}_{Q \cdot R}(R)$ maps injectively into the group of outer automorphisms of R .

Consequently, we get $\bar{N}_{Q \cdot R}(R) = \{1\}$ which implies that $Q \cdot R = R$, so that $Q \subset R$; finally, once again it follows from [2, Theorem 1.8] and from 3.1 above that $Q_\delta \subset R_\varepsilon$. Since any Fitting pointed group is a radical, the last statement is now clear. We are done.

Corollary 4.3. *Let \hat{G} be a finite k^* -group and P_γ a maximal local pointed group on $k_*\hat{G}$. A radical pointed group Q_δ on $k_*\hat{G}$ contained in P_γ is a Fitting pointed group of \hat{G} if and only if it is contained in each radical pointed group on $k_*\hat{G}$ contained in P_γ .*

Proof: It follows from Proposition 4.2 that this condition is necessary. Conversely, if Q_δ is contained in any radical pointed group on $k_*\hat{G}$ contained in P_γ , it follows from [14, Theorem A.9] that, in particular, Q_δ is contained in each *essential* pointed group R_ε contained in P_γ ; moreover, for any $x \in G$ normalizing either R_ε or P_γ , $(Q_\delta)^x$ is a Fitting pointed group on $k_*\hat{G}$ contained in P_γ and therefore it coincides with Q_δ ; hence, $N_{\hat{G}}(Q_\delta)$ contains $N_{\hat{G}}(P_\gamma)$ and $N_{\hat{G}}(R_\varepsilon)$ for each *essential* pointed group R_ε contained in P_γ . At this point, condition 4.1.1 follows from [14, Corollary A.12].

4.4. From now on, we assume that \hat{G} is a p -solvable finite k^* -group and let b be a block of \hat{G} and P_γ a maximal local pointed group on $k_*\hat{G}b$; it follows from [16, Theorem 4.6] that there exists a P -source pair (S, \hat{L}) , unique up to isomorphisms, formed by a *primitive Dade P -algebra* S and by a p -solvable finite k^* -group \hat{L} containing P , which fulfills the following two conditions

4.4.1. $C_L(\mathbb{O}_p(L)) = Z(\mathbb{O}_p(L))$ where L denotes the k^* -quotient of \hat{L} .

4.4.2. There is a P -interior algebra embedding $e_\gamma : (k_*\hat{G})_\gamma \longrightarrow S \otimes_k k_*\hat{L}$.

Note that, according to isomorphism 2.11.1, any p -subgroup of L containing $\mathbb{O}_p(L)$ has a unique local point on $k_*\hat{L}$ — actually, it coincides with $\{1\}$ (cf. 2.10). In particular, P has a unique local point $\dot{\gamma} = \{1\}$ on $k_*\hat{L}$ and therefore it follows from [12, Theorem 5.3] that it has also a unique local point $S \times \dot{\gamma}$ on $S \otimes_k k_*\hat{L}$; then, the embedding above is equivalent to the existence of a P -interior algebra isomorphism

$$(k_*\hat{G})_\gamma \cong (S \otimes_k k_*\hat{L})_{S \times \dot{\gamma}} \quad 4.4.3.$$

4.5. In particular, from isomorphism 4.4.3 and from [12, Theorem 5.3], any local pointed group Q_δ on $k_*\hat{G}$ contained in P_γ determines a local pointed group Q_δ on $k_*\hat{L}$ and this correspondence is bijective. Moreover, since we have P -interior algebra embeddings

$$k_*\hat{L} \longrightarrow S^\circ \otimes_k S \otimes_k k_*\hat{L} \longleftarrow S^\circ \otimes_k (k_*\hat{G})_\gamma \quad 4.5.1$$

and $(S^\circ \otimes_k S) \times \dot{\gamma}$ is the unique local point of P on $S^\circ \otimes_k S \otimes_k k_*\hat{L}$, we still have a P -interior algebra embedding

$$e_\gamma^\circ : k_*\hat{L} \longrightarrow S^\circ \otimes_k (k_*\hat{G})_\gamma \quad 4.5.2$$

inducing the same bijection between the sets of local pointed groups on $(k_*\hat{G})_\gamma$ and on $k_*\hat{L}$; then, since $(k_*\hat{G})_\gamma$ and $k_*\hat{L}$ admit $P \times P$ -stable bases

by the multiplication on both sides, where $P \times \{1\}$ and $\{1\} \times P$ act freely, it follows from 2.17 above applied twice that we have

$$F_{k_*\hat{G}}(R_\varepsilon, Q_\delta) = F_{k_*\hat{L}}(R_\varepsilon, Q_\delta) \subset F_S(R, Q) \quad 4.5.3$$

for any pair of local pointed groups Q_δ and R_ε on $k_*\hat{G}$ contained in P_γ , and that the choice of a *polarization* ω and of the embedding e_γ determine k^* -isomorphisms (cf. 2.8.3 and 2.17.3)

$$\hat{F}_{k_*\hat{G}}(Q_\delta) \xrightarrow{\hat{F}_{\bar{e}_\gamma}(Q_\delta)} \hat{F}_{S \otimes_k k_*\hat{L}}(Q_{S \times \delta}) \xrightarrow{\Phi_S^\omega(Q_\delta)} \hat{F}_{k_*\hat{L}}(Q_\delta) \quad 4.5.4.$$

4.6. Set $O = \mathbb{O}_p(L)$ and denote by $\dot{\eta}$ and by η the respective unique local points of O on $k_*\hat{L}$ and on $(k_*\hat{G})_\gamma$; since $O_{\dot{\eta}}$ is clearly a *Fitting pointed group* of \hat{L} , it follows from 4.5 above that O_η is a *Fitting pointed group* of \hat{G} . Moreover, from the k^* -group homomorphism 2.8.1 and from the last statement in 2.10, we get the k^* -group isomorphism

$$\hat{L}/O \cong \hat{F}_{k_*\hat{L}}(O_{\dot{\eta}}) \quad 4.6.1$$

and therefore the choice of a *polarization* ω determines a k^* -isomorphism

$$\hat{L}/O \cong \hat{F}_{k_*\hat{G}}(O_\eta) \quad 4.6.2.$$

Remark 4.7. It follows from [16, 4.7] that the Dade P -algebra S above always come from a suitable *nilpotent block* admitting P as a *defect group* and therefore, according to [15, Theorem 7.8], the *similarity* class of S in the *Dade group* $\mathcal{D}_k(P)$ is a torsion element (cf. 2.13). In particular, we can restrict our *polarizations* to the *full* subalgebra $\mathfrak{D}_k^{\text{tor}}$ of \mathfrak{D}_k over the objects (P, S) fulfilling this condition.

5. The key parameterizations

5.1. Let \hat{G} be a p -solvable finite k^* -group, b a block of \hat{G} and P_γ a maximal local pointed group on $k_*\hat{G}b$, and denote by (S, \hat{L}) a P -source pair of this block and by O_η the *Fitting pointed group* of \hat{G} contained in P_γ ; in this section, our purpose is to show that the choice of a *polarization* ω determines two bijections

$$\begin{aligned} \Gamma_{(\hat{G}, b)}^\omega : \text{Irr}_k(\hat{G}, b) &\cong \text{Irr}_k(\hat{F}_{k_*\hat{G}}(O_\eta)) \\ \Delta_{(\hat{G}, b)}^\omega : \text{Wgt}_k(\hat{G}, b) &\cong \text{Wgt}_k(\hat{F}_{k_*\hat{G}}(O_\eta)) \end{aligned} \quad 5.1.1$$

which are *natural* with respect to the isomorphisms between blocks. We first need to know the group of *exterior automorphisms* $\text{Out}_P((k_*\hat{G})_\gamma)$ (cf. 2.4) of the P -interior algebra $(k_*\hat{G})_\gamma$; recall that, according to [11, Proposition 14.9],

we have an injective group homomorphism

$$\mathrm{Out}_P((k_*\hat{G})_\gamma) \longrightarrow \mathrm{Hom}(F_{k_*\hat{G}}(P_\gamma), k^*) \quad 5.1.2$$

and therefore $\mathrm{Out}_P((k_*\hat{G})_\gamma)$ is Abelian.

Proposition 5.2. *With the notation above, there are group isomorphisms*

$$\mathrm{Out}_P((k_*\hat{G})_\gamma) \cong \mathrm{Out}_P(k_*\hat{L}) \cong \mathrm{Hom}(L, k^*) \quad 5.2.1$$

mapping $\tilde{\sigma} \in \mathrm{Out}_P((k_*\hat{G})_\gamma)$ on an element $\dot{\sigma} \in \mathrm{Out}_P(k_*\hat{L})$ such that, for any P -interior algebra embedding $e_\gamma : (k_*\hat{G})_\gamma \rightarrow S \otimes_k k_*\hat{L}$ we have

$$\tilde{e}_\gamma \circ \tilde{\sigma} = (\tilde{\mathrm{id}}_S \otimes \dot{\sigma}) \circ \tilde{e}_\gamma \quad 5.2.2,$$

and mapping $\zeta \in \mathrm{Hom}(L, k^*)$ on the exterior class of the P -interior algebra automorphism of $k_*\hat{L}$ sending $\hat{y} \in \hat{L}$ to $\zeta(y) \cdot \hat{y}$ where y is the image of \hat{y} in L . Moreover, $\mathrm{Out}_P((k_*\hat{G})_\gamma)$ acts regularly over the set of exterior embeddings from $(k_*\hat{G})_\gamma$ to $S \otimes_k k_*\hat{L}$.

Proof: Since $S^\circ \times \gamma$ is the unique local point of P on $S^\circ \otimes_k (k_*\hat{G})_\gamma$, embedding 4.5.2 induces a P -interior algebra isomorphism

$$k_*\hat{L} \cong (S^\circ \otimes_k (k_*\hat{G})_\gamma)_{S^\circ \times \gamma} \quad 5.2.3$$

and therefore, for a representative σ of $\tilde{\sigma}$, the automorphism $\mathrm{id}_S \otimes \sigma$ of $S^\circ \otimes_k (k_*\hat{G})_\gamma$, composed with a suitable inner automorphism, induces an automorphism $\dot{\sigma}$ of $k_*\hat{L}$ and it is quite clear that the exterior class $\dot{\sigma}$ of $\dot{\sigma}$ does not depend on our choices, and fulfills

$$\tilde{e}_\gamma^\circ \circ \dot{\sigma} = (\tilde{\mathrm{id}}_S \otimes \tilde{\sigma}) \circ \tilde{e}_\gamma^\circ \quad 5.2.4.$$

Tensoring embedding 4.5.2 by S and arguing as in 4.5 above, it is not difficult to prove that equality 5.2.2 also holds. Similarly, since this correspondence comes from “conjugation” via the exterior class of isomorphisms 4.4.3 and 5.2.3, it is clear that it is a group isomorphism; actually, this argument also proves the last statement.

On the other hand, for any $\zeta \in \mathrm{Hom}(L, k^*)$, it is clear that the map sending $\hat{y} \in \hat{L}$ to $\zeta(y) \cdot \hat{y}$ defines an automorphism of the k^* -group \hat{L} inducing the identity on P and thus, it determines a P -interior algebra automorphism of $k_*\hat{L}$; moreover, since y is also the image of $\zeta(y) \cdot \hat{y}$ in L , we clearly get a group homomorphism

$$\mathrm{Hom}(L, k^*) \longrightarrow \mathrm{Aut}_P(k_*\hat{L}) \quad 5.2.5.$$

Conversely, any P -interior algebra automorphism $\dot{\sigma}$ of $k_*\hat{L}$ stabilizes the *Fitting pointed group* $O_{\dot{\eta}}$, acting trivially on O ; hence, it acts on the k^* -group $\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})$ acting trivially on its k^* -quotient $F_{k_*\hat{L}}(O_{\dot{\eta}}) \subset \text{Out}(O)$ and therefore, according to isomorphism 4.6.1 above, it determines an element of

$$\text{Hom}(L/O, k^*) = \text{Hom}(L, k^*) \quad 5.2.6;$$

clearly, any inner P -interior algebra automorphism of $k_*\hat{L}$ determines the trivial element of $\text{Hom}(L, k^*)$ and thus, we easily get the second isomorphism in 5.2.1.

5.3. We are ready to define the first bijection in 5.1.1. Since the restriction determines a *Morita equivalence* between the k -algebras $k_*\hat{G}b$ and $(k_*\hat{G})_{\gamma}$ (cf. 2.7), we certainly have a *natural* bijection (cf. 2.10)

$$\text{Irr}_k(\hat{G}, b) \cong \mathcal{P}((k_*\hat{G})_{\gamma}) \quad 5.3.1$$

and any embedding $e_{\gamma} : (k_*\hat{G})_{\gamma} \rightarrow S \otimes_k k_*\hat{L}$ induces an injective map and a k^* -group isomorphism (cf. 2.5 and 2.8.3)

$$\begin{aligned} \mathcal{P}(\tilde{e}_{\gamma}) : \mathcal{P}((k_*\hat{G})_{\gamma}) &\longrightarrow \mathcal{P}(S \otimes_k k_*\hat{L}) \\ \hat{F}_{\tilde{e}_{\gamma}}(O_{\eta}) : \hat{F}_{k_*\hat{G}}(O_{\eta}) &\cong \hat{F}_{S \otimes_k k_*\hat{L}}(O_{S \times \dot{\eta}}) \end{aligned} \quad 5.3.2;$$

then, the existence of embedding 4.5.2 proves that the map $\mathcal{P}(e_{\gamma})$ is actually bijective. On the other hand, the choice of a *polarization* ω determines a k^* -group isomorphism

$$\Phi_S^{\omega}(O_{\dot{\eta}}) : \hat{F}_{S \otimes_k k_*\hat{L}}(O_{S \times \dot{\eta}}) \cong \hat{F}_{k_*\hat{L}}(O_{\dot{\eta}}) \quad 5.3.3.$$

Finally, isomorphism 4.6.1 determines a canonical bijection

$$\Gamma_{\hat{L}} : \text{Irr}_k(\hat{L}) \cong \text{Irr}_k(\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})) \quad 5.3.4.$$

Corollary 5.4. *With the notation and the choice above, there is a bijection*

$$\Gamma_{(\hat{G}, b)}^{\omega} : \text{Irr}_k(\hat{G}, b) \cong \text{Irr}_k(\hat{F}_{k_*\hat{G}}(O_{\eta})) \quad 5.4.1$$

such that, for any embedding $e_{\gamma} : (k_\hat{G})_{\gamma} \rightarrow S \otimes_k k_*\hat{L}$, we have the commutative diagram*

$$\begin{array}{ccccc} \text{Irr}_k(\hat{G}, b) & \cong & \mathcal{P}(S \otimes_k k_*\hat{L}) & \cong & \text{Irr}_k(\hat{L}) \\ \Gamma_{(\hat{G}, b)}^{\omega} \wr & & & & \Gamma_{\hat{L}} \wr \\ \text{Irr}_k(\hat{F}_{k_*\hat{G}}(O_{\eta})) & \cong & \text{Irr}_k(\hat{F}_{S \otimes_k k_*\hat{L}}(O_{S \times \dot{\eta}})) & \cong & \text{Irr}_k(\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})) \end{array} \quad 5.4.2.$$

Proof: It is clear that, for a choice of an embedding

$$e_{\gamma} : (k_*\hat{G})_{\gamma} \longrightarrow S \otimes_k k_*\hat{L} \quad 5.4.3,$$

the bijections 5.3.1 and $\mathcal{P}(\tilde{e}_\gamma)$, and the k^* -group isomorphism $\hat{F}_{\tilde{e}_\gamma}(O_\eta)$ above determine the horizontal left-hand bijections in diagram 5.4.2; the top horizontal right-hand bijection follow from 2.10 and 2.17, and the bottom horizontal right-hand bijection from isomorphism 5.3.3 up to the choice of ω ; then, the bijection $\Gamma_{\hat{L}}$ and the commutativity of the diagram define the bijection $\Gamma_{(\hat{G}, b)}^\omega$.

We claim that this bijection does not depend on the choice of e_γ ; indeed, for another choice e'_γ of this embedding, it follows from Proposition 5.2 that there is $\tilde{\sigma} \in \text{Out}_P((k_* \hat{G})_\gamma)$ fulfilling

$$\tilde{e}'_\gamma = \tilde{e}_\gamma \circ \tilde{\sigma} = (\text{id}_S \otimes \dot{\tilde{\sigma}}) \circ \tilde{e}_\gamma \quad 5.4.4$$

and therefore, with obvious notation, we get the following commutative diagrams

$$\begin{array}{ccc} \text{Irr}_k(\hat{G}, b) & \cong & \mathcal{P}((k_* \hat{G})_\gamma) \xrightarrow{\mathcal{P}(\tilde{e}'_\gamma)} \mathcal{P}(S \otimes_k k_* \hat{L}) \\ \parallel & & \parallel \wr \mathcal{P}(\text{id}_S \otimes \dot{\tilde{\sigma}}) \\ \text{Irr}_k(\hat{G}, b) & \cong & \mathcal{P}((k_* \hat{G})_\gamma) \xrightarrow{\mathcal{P}(\tilde{e}_\gamma)} \mathcal{P}(S \otimes_k k_* \hat{L}) \end{array} \quad 5.4.5$$

$$\begin{array}{ccc} \text{Irr}_k(\hat{F}_{k_* \hat{G}}(O_\eta)) & \xrightarrow{\text{Irr}_k(\hat{F}_{\tilde{e}'_\gamma}(O_\eta))} & \text{Irr}_k(\hat{F}_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}})) \\ \parallel & & \parallel \wr \text{Irr}_k(\hat{F}_{\text{id}_S \otimes \dot{\tilde{\sigma}}}(O_{S \times \dot{\eta}})) \\ \text{Irr}_k(\hat{F}_{k_* \hat{G}}(O_\eta)) & \xrightarrow{\text{Irr}_k(\hat{F}_{\tilde{e}_\gamma}(O_\eta))} & \text{Irr}_k(\hat{F}_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}})) \end{array} \quad 5.4.6$$

Moreover, we have the evident commutative diagram

$$\begin{array}{ccc} \mathcal{P}(S \otimes_k k_* \hat{L}) & \cong & \text{Irr}_k(\hat{L}) \\ \mathcal{P}(\text{id}_S \otimes \dot{\tilde{\sigma}}) \wr & & \wr \text{Irr}_k(\dot{\tilde{\sigma}}) \\ \mathcal{P}(S \otimes_k k_* \hat{L}) & \cong & \text{Irr}_k(\hat{L}) \end{array} \quad 5.4.7;$$

on the other hand, since the groups of k^* -group automorphisms of $\hat{F}_{k_* \hat{L}}(O_{\dot{\eta}})$ and $\hat{F}_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}})$ which induce the identity over (cf. 2.17 applied twice)

$$F_{k_* \hat{L}}(O_{\dot{\eta}}) = F_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}}) \quad 5.4.8$$

are both canonically isomorphic to the Abelian group $\text{Hom}(F_{k_* \hat{L}}(O_{\dot{\eta}}), k^*)$, we still have the commutative diagram

$$\begin{array}{ccc} \text{Irr}_k(\hat{F}_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}})) & \cong & \text{Irr}_k(\hat{F}_{k_* \hat{L}}(O_{\dot{\eta}})) \\ \text{Irr}_k(\hat{F}_{\text{id}_S \otimes \dot{\tilde{\sigma}}}(O_{S \times \dot{\eta}})) \wr & & \wr \text{Irr}_k(\hat{F}_{\dot{\tilde{\sigma}}}(O_{\dot{\eta}})) \\ \text{Irr}_k(\hat{F}_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}})) & \cong & \text{Irr}_k(\hat{F}_{k_* \hat{L}}(O_{\dot{\eta}})) \end{array} \quad 5.4.9.$$

Finally, from isomorphism 4.6.2 we obviously get the following commutative diagram

$$\begin{array}{ccc}
\text{Irr}_k(\hat{L}) & \xrightarrow{\text{Irr}_k(\dot{\sigma})} & \text{Irr}_k(\hat{L}) \\
\Gamma_{\hat{L}} \wr \parallel & & \wr \parallel \Gamma_{\hat{L}} \\
\text{Irr}_k(\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})) & \xrightarrow{\text{Irr}_k(\hat{F}_{\hat{\sigma}}(O_{\hat{\eta}}))} & \text{Irr}_k(\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}}))
\end{array} \quad 5.4.10;$$

now, our claim follows from putting together all these commutative diagrams.

5.5. In order to define the second bijection in 5.1.1, let (R_ε, X) be a b -weight of \hat{G} ; for our purposes, we may assume that P_γ contains R_ε ; then, R_ε and $R_{\dot{\varepsilon}}$ respectively contain O_η and $O_{\hat{\eta}}$; recall that, with the notation and the choice in 4.5 above, we have k^* -group isomorphisms

$$\hat{F}_{k_*\hat{G}}(R_\varepsilon) \xrightarrow{\hat{F}_{\varepsilon\gamma}(R_\varepsilon)} \hat{F}_{S \otimes_k k_*\hat{L}}(R_{S \times \dot{\varepsilon}}) \xrightarrow{\Phi_S^\omega(R_{\dot{\varepsilon}})} \hat{F}_{k_*\hat{L}}(R_{\dot{\varepsilon}}) \quad 5.5.1$$

and, in this case, X determines an isomorphism class \dot{X} of *simple projective* $k_*\hat{F}_{k_*\hat{L}}(R_{\dot{\varepsilon}})$ -modules; moreover, we clearly have $N_{\hat{L}}(R_{\dot{\varepsilon}}) = N_{\hat{L}}(R)$ and from 2.11.2 it is easily checked that the k^* -group homomorphism 2.8.3 induces a k^* -group isomorphism

$$\bar{N}_{\hat{L}}(R_{\dot{\varepsilon}}) \cong \hat{F}_{k_*\hat{L}}(R_{\dot{\varepsilon}}) \quad 5.5.2;$$

consequently, the pair (R, \dot{X}) is a *weight* of \hat{L} .

Proposition 5.6. *With the notation and the choice above, let (R_ε, X) and $(R'_{\varepsilon'}, X')$ be b -weights of \hat{G} such that P_γ contains R_ε and $R'_{\varepsilon'}$. If (R_ε, X) and $(R'_{\varepsilon'}, X')$ are G -conjugate then the corresponding weights (R, \dot{X}) and (R', \dot{X}') of \hat{L} are L -conjugate. In particular, this correspondence induces a bijection*

$$\text{Wgt}_k^\omega(e_\gamma) : \text{Wgt}_k(\hat{G}, b) \cong \text{Wgt}_k(\hat{L}) \quad 5.6.1.$$

Proof: Assume that $(R'_{\varepsilon'}, X')^x = (R_\varepsilon, X)$ for some $x \in G$; then, the conjugation by x determines a $(k_*\hat{G})_\gamma$ -fusion φ from R_ε to $R'_{\varepsilon'}$ (cf. 2.6), and the corresponding R -interior algebra isomorphism

$$f_\varphi : (k_*\hat{G})_\varepsilon \cong \text{Res}_\varphi((k_*\hat{G})_{\varepsilon'}) \quad 5.6.2$$

induces a k^* -group isomorphism (cf 2.8.3)

$$\hat{F}_{\hat{f}_\varphi}(R_\varepsilon) : \hat{F}_{(k_*\hat{G})_\varepsilon}(R_\varepsilon) \cong \hat{F}_{(k_*\hat{G})_{\varepsilon'}}(R'_{\varepsilon'}) \quad 5.6.3;$$

actually, we have $X = \text{Res}_{\hat{F}_{\hat{f}_\varphi}(R_\varepsilon)}(X')$.

But, according to equality 4.5.3, the group homomorphism φ is also a $k_*\hat{L}$ -fusion from $R_{\hat{\varepsilon}}$ to $R'_{\hat{\varepsilon}'}$, so that $\varphi: R \cong R'$ is also induced by some element $\dot{x} \in L$ (cf. statement 2.11.2); moreover, we have the corresponding R -interior algebra isomorphism

$$\dot{f}_{\varphi} : (k_*\hat{L})_{\hat{\varepsilon}} \cong \text{Res}_{\varphi}((k_*\hat{L})_{\hat{\varepsilon}'}) \quad 5.6.4$$

inducing a k^* -group isomorphism (cf 2.8.3)

$$\hat{F}_{\dot{f}_{\varphi}}(R_{\hat{\varepsilon}}) : \hat{F}_{(k_*\hat{L})_{\hat{\varepsilon}}}(R_{\hat{\varepsilon}}) \cong \hat{F}_{(k_*\hat{L})_{\hat{\varepsilon}'}}(R'_{\hat{\varepsilon}'}) \quad 5.6.5.$$

Then, the commutativity of diagrams 2.9.5 and 2.17.4 applied here yields the following commutative diagrams of k^* -group isomorphisms

$$\begin{array}{ccccc} \hat{F}_{(k_*\hat{G})_{\gamma}}(R_{\varepsilon}) & \xrightarrow{\hat{F}_{\varepsilon\gamma}(R_{\varepsilon})} & \hat{F}_{S \otimes_k k_*\hat{L}}(R_{S \times \varepsilon}) & \xrightarrow{\Phi_S^{\omega}(R_{\varepsilon})} & \hat{F}_{k_*\hat{L}}(R_{\hat{\varepsilon}}) \\ \hat{F}_{\dot{f}_{\varphi}}(R_{\varepsilon}) \wr & & \wr & & \wr \hat{F}_{\dot{f}_{\varphi}}(R_{\hat{\varepsilon}}) \\ \hat{F}_{(k_*\hat{G})_{\gamma}}(R'_{\varepsilon'}) & \xrightarrow{\hat{F}_{\varepsilon'\gamma}(R'_{\varepsilon'})} & \hat{F}_{S \otimes_k k_*\hat{L}}(R'_{S \times \varepsilon'}) & \xrightarrow{\Phi_S^{\omega}(R'_{\varepsilon'})} & \hat{F}_{k_*\hat{L}}(R'_{\hat{\varepsilon}'}) \end{array} \quad 5.6.6.$$

Consequently, we also have $\dot{X} = \text{Res}_{\hat{F}_{\dot{f}_{\varphi}}(R_{\hat{\varepsilon}})}(\dot{X}')$ and therefore we get

$$(R'_{\hat{\varepsilon}'}, \dot{X}')^{\dot{x}} = (R_{\hat{\varepsilon}}, \dot{X}) \quad 5.6.7;$$

that is to say, the correspondence above induces a map

$$\text{Wgt}_k^{\omega}(e_{\gamma}) : \text{Wgt}_k(\hat{G}, b) \longrightarrow \text{Wgt}_k(\hat{L}) \quad 5.6.8.$$

which is quite clear that it is a bijection. We are done.

Proposition 5.7. *With the notation above, the canonical k^* -group isomorphism $\hat{L}/O \cong \hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})$ induces a bijection*

$$\Delta_{\hat{L}} : \text{Wgt}_k(\hat{L}) \cong \text{Wgt}_k(\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})) \quad 5.7.1.$$

Proof: Let (R, Y) be a *weight* of \hat{L} ; since the unity element in $k_*\hat{L}$ is a block of \hat{L} and condition 4.4.1 holds, R has a unique local point ε on $k_*\hat{L}$ and R_{ε} is a radical pointed group which contains $O_{\hat{\eta}}$ (cf. Corollary 4.3); moreover, since we have the k^* -group isomorphism $\hat{L}/O \cong \hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})$ (cf. 4.6.2), setting $\bar{R} = R/O$ and identifying \bar{R} with its image in $\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})$, the normalizer $N_{\hat{L}}(R_{\varepsilon}) = N_{\hat{L}}(R)$ is just the converse image in \hat{L} of $N_{\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})}(\bar{R})$ and therefore we have the canonical k^* -group isomorphism

$$\bar{N}_{\hat{L}}(R) \cong \bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})}(\bar{R}) \quad 5.7.2;$$

in particular, Y determines an isomorphism class \bar{Y} of simple $N_{\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})}(\bar{R})$ -modules of vertex \bar{R} , so that the pair (\bar{R}, \bar{Y}) is a *weight* of $\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})$.

Conversely, if we start with a *weight* (\bar{R}, \bar{Y}) of $\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})$, it is clear that, for the converse image R of \bar{R} in \hat{L} , isomorphism 5.6.2 still holds and therefore \bar{Y} determines an isomorphism class Y of simple $k_*\hat{L}$ -modules of vertex R , so that the pair (R, Y) is a *weight* of \hat{L} . Since this correspondence is compatible with the L -conjugation, we get the announced bijection 5.6.1.

Corollary 5.8. *With the notation and the choice above, there is a bijection*

$$\Delta_{(\hat{G}, b)}^\omega : \text{Wgt}_k(\hat{G}, b) \cong \text{Wgt}_k(\hat{F}_{k_*\hat{G}}(O_{\hat{\eta}})) \quad 5.8.1$$

such that, for any embedding $e_\gamma : (k_*\hat{G})_\gamma \rightarrow S \otimes_k k_*\hat{L}$, we have the commutative diagram

$$\begin{array}{ccc} \text{Wgt}_k(\hat{G}, b) & \xrightarrow{\text{Wgt}_k^\omega(e_\gamma)} & \text{Wgt}_k(\hat{L}) \\ \Delta_{(\hat{G}, b)}^\omega \wr \parallel & & \Delta_{\hat{L}} \wr \parallel \\ \text{Wgt}_k(\hat{F}_{k_*\hat{G}}(O_{\hat{\eta}})) & \cong & \text{Wgt}_k(\hat{F}_{k_*\hat{L}}(O_{\hat{\eta}})) \end{array} \quad 5.8.2$$

where the bottom bijection is induced by the k^* -group isomorphisms

$$\hat{F}_{k_*\hat{G}}(O_{\hat{\eta}}) \xrightarrow{\hat{F}_{\tilde{e}_\gamma}(O_{\hat{\eta}})} \hat{F}_{S \otimes_k k_*\hat{L}}(O_{S \times \hat{\eta}}) \xrightarrow{\Phi_S^\omega(O_{\hat{\eta}})} \hat{F}_{k_*\hat{L}}(O_{\hat{\eta}}) \quad 5.8.3$$

Proof: It is clear that, for a choice of e_γ , Propositions 5.6 and 5.7, and the commutativity of the diagram define the bijection $\Delta_{(\hat{G}, b)}^\omega$. We claim that this bijection does not depend on this choice; indeed, for another choice e'_γ of this embedding, it follows from Proposition 5.2 that there is $\tilde{\sigma} \in \text{Out}_P((k_*\hat{G})_\gamma)$ fulfilling

$$\tilde{e}'_\gamma = \tilde{e}_\gamma \circ \tilde{\sigma} = (\tilde{\text{id}}_S \otimes \tilde{\sigma}) \circ \tilde{e}_\gamma \quad 5.8.4$$

in particular, if (R_ε, X) is a b -weight of \hat{G} and $(R_{\hat{\varepsilon}}, \dot{X})$ the corresponding *weight* of \hat{L} in 5.5 above — \dot{X} is the isomorphism class of a *simple projective* $k_*\hat{F}_{k_*\hat{L}}(R_{\hat{\varepsilon}})$ -module V restricted to $N_{\hat{L}}(R)$ — then $\text{Wgt}_k^\omega(e'_\gamma)$ sends the G -conjugacy class of (R_ε, X) to the L -conjugacy class of $(R_{\hat{\varepsilon}}, \dot{X}')$ where \dot{X}' is the isomorphism class of corresponding the *simple projective* $k_*\hat{F}_{k_*\hat{L}}(R_{\hat{\varepsilon}})$ -module $\text{Res}_{\hat{F}_{\hat{\sigma}}^{-1}(R_{\hat{\varepsilon}})}(V)$, since $\text{Hom}(L, k^*)$ clearly acts trivially on the set of local pointed groups on $k_*\hat{L}$ and we have the commutative diagram (cf. 2.17.4)

$$\begin{array}{ccc} \hat{F}_{S \otimes_k k_*\hat{L}}(R_{S \times \hat{\varepsilon}}) & \xrightarrow{\Phi_S^\omega(R_{\hat{\varepsilon}})} & \hat{F}_{k_*\hat{L}}(R_{\hat{\varepsilon}}) \\ \hat{F}_{\tilde{\text{id}}_S \otimes \tilde{\sigma}}(R_{S \times \hat{\varepsilon}}) \wr \parallel & & \wr \parallel \hat{F}_{\hat{\sigma}}(R_{\hat{\varepsilon}}) \\ \hat{F}_{S \otimes_k k_*\hat{L}}(R_{S \times \hat{\varepsilon}}) & \xrightarrow{\Phi_S^\omega(R_{\hat{\varepsilon}})} & \hat{F}_{k_*\hat{L}}(R_{\hat{\varepsilon}}) \end{array} \quad 5.8.5$$

Now, setting $\bar{R} = R/O$, since we have (cf. isomorphisms 4.6.1 and 5.5.2)

$$\hat{F}_{k_*\hat{L}}(R_{\dot{\varepsilon}}) \cong \bar{N}_{\hat{L}}(R_{\dot{\varepsilon}}) = \bar{N}_{\hat{L}}(R) \cong \bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})}(\bar{R}) \quad 5.8.6,$$

V determines a *simple projective* $\bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})}(\bar{R})$ -module \bar{V} ; moreover, since $\text{Hom}(L, k^*)$ clearly stabilizes \hat{L} and it acts trivially on P , the corresponding representative $\dot{\sigma}$ of $\dot{\sigma}$ induces a k^* -group automorphism $\dot{\sigma}$ of $\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})$ (cf. isomorphism 4.6.1) which stabilizes $\bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})}(\bar{R})$, and it is quite clear that, with obvious notation, we get the following commutative diagram

$$\begin{array}{ccc} \hat{F}_{k_*\hat{L}}(R_{\dot{\varepsilon}}) & \cong & \bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})}(\bar{R}) \\ \hat{F}_{\dot{\sigma}}(R_{\dot{\varepsilon}}) \wr & & \wr \bar{N}_{\dot{\sigma}}(\bar{R}) \\ \hat{F}_{k_*\hat{L}}(R_{\dot{\varepsilon}}) & \cong & \bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})}(\bar{R}) \end{array} \quad 5.8.7;$$

hence, *via* isomorphisms 5.6.7, $\text{Res}_{\hat{F}_{\dot{\sigma}-1}(R_{\dot{\varepsilon}})}(V)$ determines the *simple projective* $\bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})}(\bar{R})$ -module $\text{Res}_{\bar{N}_{\dot{\sigma}-1}(\bar{R})}(\bar{V})$.

At this point, denoting by \dot{X} and \dot{X}' the respective isomorphism classes of the $\bar{N}_{\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})}(\bar{R})$ -modules \bar{V} and $\text{Res}_{\bar{N}_{\dot{\sigma}-1}(\bar{R})}(\bar{V})$, it follows from Proposition 5.7 that $\Delta_{\hat{L}}$ maps $(R_{\dot{\varepsilon}}, \dot{X})$ on (\bar{R}, \dot{X}) , and $(R_{\dot{\varepsilon}}, \dot{X}')$ on (\bar{R}, \dot{X}') . But, we also have the following commutative diagram (cf. 2.17.4)

$$\begin{array}{ccc} \hat{F}_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}}) & \xrightarrow{\Phi_S^{\omega}(O_{\dot{\eta}})} & \hat{F}_{k_*\hat{L}}(O_{\dot{\eta}}) \\ \hat{F}_{\text{id}_S \otimes \dot{\sigma}}(O_{S \times \dot{\eta}}) \wr & & \wr \hat{F}_{\dot{\sigma}}(O_{\dot{\eta}}) \\ \hat{F}_{S \otimes_k k_* \hat{L}}(O_{S \times \dot{\eta}}) & \xrightarrow{\Phi_S^{\omega}(O_{\dot{\eta}})} & \hat{F}_{k_*\hat{L}}(O_{\dot{\eta}}) \end{array} \quad 5.8.8$$

and we consider its restriction to all the normalizers of \bar{R} . Consequently, since we have (cf. 5.8.4)

$$\hat{F}_{\bar{e}'_{\gamma}}(O_{\eta}) = \hat{F}_{\text{id}_S \otimes \dot{\sigma}}(O_{S \times \dot{\eta}}) \circ \hat{F}_{\bar{e}_{\gamma}}(O_{\eta}) \quad 5.8.9,$$

the corresponding bottom bijections in diagram 5.8.2 maps the *weights* (\bar{R}, \dot{X}) and (\bar{R}, \dot{X}') of $\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})$ on the *same weight* of $\hat{F}_{k_*\hat{L}}(O_{\dot{\eta}})$. We are done.

6. The Fitting block sequences

6.1. In order to exhibit bijections between the sets of isomorphism classes of simple $k_*\hat{G}$ -modules and of G -conjugacy classes of *weights* of \hat{G} , we need a third set, namely the set of G -conjugacy classes of *Fitting block sequences* of \hat{G} . We call *Fitting block sequence* of \hat{G} any sequence $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ of pairs formed by a k^* -group \hat{G}_n and by a block b_n of \hat{G}_n , such that $\hat{G}_0 = \hat{G}$ and that, for any $n \in \mathbb{N}$, we have $\hat{G}_{n+1} = \hat{F}_{k_*\hat{G}_n}(O_{\eta_n}^n)$ for some *Fitting pointed*

group $O_{\eta_n}^n$ of \hat{G}_n . Note that, since clearly $|G_{n+1}| \leq |G_n|$, such a sequence stabilizes, and actually we have $|G_{n+1}| = |G_n|$ if and only if b_n is a block of defect zero of \hat{G}_n (cf. statement 4.4.1). Moreover, for any $h \in \mathbb{N}$, the sequence $\mathcal{B}_h = \{(\hat{G}_{h+n}, b_{h+n})\}_{n \in \mathbb{N}}$ is clearly a *Fitting block sequence* of \hat{G}_h .

6.2. If \hat{G}' is a k^* -group isomorphic to \hat{G} and $\theta: \hat{G} \cong \hat{G}'$ a k^* -group isomorphism of \hat{G} , it is quite clear that, from any *Fitting block sequence* $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ of \hat{G} , we are able to construct a *Fitting block sequence* $\mathcal{B}' = \{(\hat{G}'_n, b'_n)\}_{n \in \mathbb{N}}$ of \hat{G}' inductively defining a sequence of k^* -group isomorphisms $\theta_n: \hat{G}_n \cong \hat{G}'_n$ by $\theta_0 = \theta$ and, for any $n \in \mathbb{N}$, by (cf. 2.9)

$$\theta_{n+1} = \hat{F}_{\theta_n}(O_{\eta_n}^n) : \hat{F}_{k_* \hat{G}_n}(O_{\eta_n}^n) \cong \hat{F}_{k_* \hat{G}'_n}(\theta_n(O^n)_{\theta_n(\eta_n)}) \quad 6.2.1,$$

where we still denote by $\theta_n: k_* \hat{G}_n \cong k_* \hat{G}'_n$ the corresponding k -algebra isomorphism, and setting

$$b'_n = \theta_n(b_n) \quad \text{and} \quad \hat{G}_{n+1} = \hat{F}_{k_* \hat{G}'_n}(\theta_n(O^n)_{\theta_n(\eta_n)}) \quad 6.2.2$$

for any $n \in \mathbb{N}$. In particular, the group of inner automorphisms of G acts on the set of *Fitting block sequences* of \hat{G} and then we denote by $\text{Fbs}_k(\hat{G})$ the set of “ G -conjugacy classes” of the *Fitting block sequences* of \hat{G} , and by $N_G(\mathcal{B})$ the stabilizer of \mathcal{B} in G .

6.3. In this section, our purpose is to show that the choice of a *polarization* ω determines two bijections

$$\text{Fbs}_k(\hat{G}) \cong \text{Irr}_k(\hat{G}) \quad \text{and} \quad \text{Fbs}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G}) \quad 6.3.1$$

which are *natural* with respect to the k^* -group isomorphisms, the composition of the inverse of the first one with the second one being our announced parameterization.

6.4. Let $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ be a *Fitting block sequence* of \hat{G} , so that we have $\hat{G}_{n+1} = \hat{F}_{k_* \hat{G}_n}(O_{\eta_n}^n)$ for some Fitting pointed group $O_{\eta_n}^n$ of \hat{G}_n and, choosing a *polarization* ω , we denote by

$$\begin{aligned} \Gamma_{(\hat{G}_n, b_n)}^\omega &: \text{Irr}_k(\hat{G}_n, b_n) \cong \text{Irr}_k(\hat{G}_{n+1}) \\ \Delta_{(\hat{G}_n, b_n)}^\omega &: \text{Wgt}_k(\hat{G}_n, b_n) \cong \text{Wgt}_k(\hat{G}_{n+1}) \end{aligned} \quad 6.4.1$$

the bijections coming from Corollaries 5.4 and 5.8. Let us call *character sequence ω -associated to \mathcal{B}* any sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ where φ_n belongs to $\text{Irr}_k(\hat{G}_n, b_n)$ in such a way that we have

$$\Gamma_{(\hat{G}_n, b_n)}^\omega(\varphi_n) = \varphi_{n+1} \quad 6.4.2$$

for any $n \in \mathbb{N}$. Similarly, let us call *weight sequence* ω -associated to \mathcal{B} any sequence $\{(\overline{R^n}, \overline{Y^n})\}_{n \in \mathbb{N}}$ where $(\overline{R^n}, \overline{Y^n})$ is the G_n -conjugacy class of a *weight* (R^n, Y^n) of \hat{G}_n , determining a G_n -conjugacy class $(\overline{R_{\varepsilon_n}^n}, \overline{X^n})$ of b_n -weights of \hat{G}_n (cf. statement 3.4.1) in such a way that we have

$$\Delta_{(\hat{G}_n, b_n)}^\omega((\overline{R_{\varepsilon_n}^n}, \overline{X^n})) = (\overline{R^{n+1}}, \overline{Y^{n+1}}) \quad 6.4.3$$

for any $n \in \mathbb{N}$.

Theorem 6.5. *With the notation and the choice above, any Fitting block sequence $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ of \hat{G} admits a unique character sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ and a unique weight sequence $\{(\overline{R^n}, \overline{Y^n})\}_{n \in \mathbb{N}}$ ω -associated to \mathcal{B} . Moreover, the correspondences mapping \mathcal{B} to φ_0 and to $(\overline{R^0}, \overline{Y^0})$ induce two natural bijections*

$$\text{Fbs}_k(\hat{G}) \cong \text{Irr}_k(\hat{G}) \quad \text{and} \quad \text{Fbs}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G}) \quad 6.5.1.$$

Proof: Since the sequence \mathcal{B} stabilizes, we can argue by induction on the “length to stabilization”. If this length is zero then the block b_0 is already of defect zero and therefore $\text{Irr}_k(\hat{G}_0, b_0)$ has a unique element φ_0 and, setting $\varphi_n = \varphi_0$ for any $n \in \mathbb{N}$, we get a *character sequence* ω -associated to \mathcal{B} ; similarly, $\text{Wgt}_k(\hat{G}_0, b_0)$ has a unique element and the corresponding constant sequence defines a *weight sequence* ω -associated to \mathcal{B} .

If the “length to stabilization” is not zero then the Fitting block sequence $\mathcal{B}_1 = \{(\hat{G}_{1+n}, b_{1+n})\}_{n \in \mathbb{N}}$ of \hat{G}_1 already admits a character sequence $\{\varphi_{1+n}\}_{n \in \mathbb{N}}$ and a weight sequence $\{(\overline{R^{1+n}}, \overline{Y^{1+n}})\}_{n \in \mathbb{N}}$ ω -associated to \mathcal{B}_1 ; then, in order to get a character sequence and a weight sequence ω -associated to \mathcal{B} , it suffices to define (cf. 6.4.1)

$$\begin{aligned} \varphi_0 &= (\Gamma_{(\hat{G}_0, b_0)}^\omega)^{-1}(\varphi_1) \\ (\overline{R_{\varepsilon_0}^0}, \overline{X^0}) &= (\Delta_{(\hat{G}_0, b_0)}^\omega)^{-1}((\overline{R^1}, \overline{Y^1})) \end{aligned} \quad 6.5.2$$

and to consider the G -conjugacy class $(\overline{R^0}, \overline{Y^0})$ of *weights* of \hat{G} determined by $(\overline{R_{\varepsilon_0}^0}, \overline{X^0})$ (cf. statement 3.4.1).

On the other hand, since the maps $\Gamma_{(\hat{G}_n, b_n)}^\omega$ and $\Delta_{(\hat{G}_n, b_n)}^\omega$ are bijective, equalities 6.4.2 and 6.4.3 show that a *character sequence* $\{\varphi_n\}_{n \in \mathbb{N}}$ and a *weight sequence* $\{(\overline{R^n}, \overline{Y^n})\}_{n \in \mathbb{N}}$ ω -associated to \mathcal{B} are uniquely determined by one of their terms; but, for n big enough, we know that b_n is a block of defect zero of \hat{G}_n and then φ_n and $(\overline{R^n}, \overline{Y^n})$ are uniquely determined; consequently, $\{\varphi_n\}_{n \in \mathbb{N}}$ and $\{(\overline{R^n}, \overline{Y^n})\}_{n \in \mathbb{N}}$ are uniquely determined and it is quite clear that they only depend on the G -conjugacy class of \mathcal{B} ; thus, since $\Gamma_{(\hat{G}_n, b_n)}^\omega$ and $\Delta_{(\hat{G}_n, b_n)}^\omega$ are *natural*, we have obtained two *natural* maps

$$\text{Fbs}_k(\hat{G}) \longrightarrow \text{Irr}_k(\hat{G}) \quad \text{and} \quad \text{Fbs}_k(\hat{G}) \longrightarrow \text{Wgt}_k(\hat{G}) \quad 6.5.3.$$

We claim that they are both bijective; actually, we will define the inverse maps. For any $\varphi \in \text{Irr}_k(\hat{G})$, we inductively define two sequences $\{\varphi_n\}_{n \in \mathbb{N}}$ and $\{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ by setting $\varphi_0 = \varphi$, $\hat{G}_0 = \hat{G}$ and by denoting by b_0 the block of φ , and further, for any $n \in \mathbb{N}$, by setting

$$\varphi_{n+1} = \Gamma_{(\hat{G}_n, b_n)}^\omega(\varphi_n) \quad , \quad \hat{G}_{n+1} = \hat{F}_{k_* \hat{G}_n}(O_{\eta_n}^n) \quad 6.5.4$$

for some Fitting pointed group $O_{\eta_n}^n$ on $k_* \hat{G}_n b_n$, and by denoting by b_{n+1} the block of φ_{n+1} ; then, it is clear that $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ is a *Fitting block sequence* of \hat{G} and that $\{\varphi_n\}_{n \in \mathbb{N}}$ becomes the *character sequence* ω -associated to \mathcal{B} ; note that, our construction only depends on the choice of the Fitting pointed group $O_{\eta_n}^n$ on $k_* \hat{G}_n b_n$ for a *finite set of values of n* . Moreover, since all the *Fitting pointed group* on $k_* \hat{G}_n b_n$ are mutually G_n -conjugate (cf. Proposition 4.2), φ determines a unique G -conjugacy class of *Fitting block sequence* of \hat{G} . That is to say, we have obtained a map

$$\text{Irr}_k(\hat{G}) \longrightarrow \text{Fbs}_k(\hat{G}) \quad 6.5.5$$

and it is easily checked that it is the inverse of the left-hand map in 6.5.3.

Analogously, for any $(\overline{R}, \overline{Y}) \in \text{Wgt}_k(\hat{G})$, we inductively define two sequences $\{(\overline{R}^n, \overline{Y}^n)\}_{n \in \mathbb{N}}$ and $\{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ by setting $(\overline{R}^0, \overline{Y}^0) = (\overline{R}, \overline{Y})$, $\hat{G}_0 = \hat{G}$ and by denoting by b_0 the block of \hat{G}_0 determined by $(\overline{R}, \overline{Y})$ (cf. bijection 3.4.2), and further, for any $n \in \mathbb{N}$, by setting

$$(\overline{R}^{n+1}, \overline{Y}^{n+1}) = \Delta_{(\hat{G}_n, b_n)}^\omega((\overline{R}^n, \overline{X}^n)) \quad , \quad \hat{G}_{n+1} = \hat{F}_{k_* \hat{G}_n}(O_{\eta_n}^n) \quad 6.5.6$$

where $(\overline{R}^n, \overline{X}^n)$ is the b_n -weight of \hat{G}_n determined by $(\overline{R}^n, \overline{Y}^n)$ and $O_{\eta_n}^n$ a Fitting pointed group on $k_* \hat{G}_n b_n$, and by denoting by b_{n+1} the block of determined by the *weight* $(\overline{R}^{n+1}, \overline{Y}^{n+1})$ (cf. bijection 3.4.2); then, it is clear that $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$ is a *Fitting block sequence* of \hat{G} and that $\{(\overline{R}^n, \overline{Y}^n)\}_{n \in \mathbb{N}}$ becomes the *weight sequence* ω -associated to \mathcal{B} . As above, our construction only depends on the choice of the Fitting pointed group $O_{\eta_n}^n$ on $k_* \hat{G}_n b_n$ for a *finite set of values of n* and therefore we have obtained a map

$$\text{Wgt}_k(\hat{G}) \longrightarrow \text{Fbs}_k(\hat{G}) \quad 6.5.7$$

which is the inverse of the right-hand map in 6.5.3.

7. Vertex, sources and multiplicity modules

7.1. Let \hat{G} be again a p -solvable finite k^* -group and choose a *polarization* ω ; then, it follows from Theorem 6.5 above that any simple $k_* \hat{G}$ -module M determines a G -conjugacy class $(\overline{R}, \overline{Y})$ of *weights* of \hat{G} and in this section we discuss the relationship between this G -conjugacy class $(\overline{R}, \overline{Y})$ and the G -conjugacy class of the triples formed by a *vertex* Q , a Q -source E and a *multiplicity* $\hat{N}_G(Q)_E$ -module V of M (cf. 2.5).

7.2. Actually, M also determines a G -conjugacy class of *Fitting block sequences* $\mathcal{B} = \{(\hat{G}_n, b_n)\}_{n \in \mathbb{N}}$; let us denote by $\{O_{\eta_n}^n\}_{n \in \mathbb{N}}$ the corresponding sequence of Fitting pointed groups $O_{\eta_n}^n$ on $k_*\hat{G}_n b_n$, so that for any $n \in \mathbb{N}$ we have

$$\hat{G}_{n+1} = \hat{F}_{k_*\hat{G}}(O_{\eta_n}^n) \quad 7.2.1;$$

let $P_{\gamma_n}^n$ be a maximal local pointed group on $k_*\hat{G}_n$ containing $O_{\eta_n}^n$ and (S_n, \hat{L}_n) a P^n -source pair for $k_*\hat{G}_n b_n$ (cf. 4.4); note that, according to statement 2.11.2 above and to [12, Lemma 3.10], up to a suitable identification, $\bar{P}^n = P^n/O^n$ is a Sylow p -subgroup of G_{n+1} and therefore there is $\bar{x} \in G_{n+1}$ such that $(\bar{P}^n)^{\bar{x}}$ contains P_{n+1} ; thus, since the sequence \mathcal{B} “stabilizes”, up to finite number of choices we may assume that \bar{P}_n contains P_{n+1} for any $n \in \mathbb{N}$.

7.3. Moreover, from Theorem 6.5 we still obtain a *weight sequence* $\{(\overline{R^n}, \overline{Y^n})\}_{n \in \mathbb{N}}$ ω -associated to \mathcal{B} starting on $(\overline{R}, \overline{Y}) = (\overline{R^0}, \overline{Y^0})$, and from Corollary 5.4 we get a *simple sequence* $\{M_n\}_{n \in \mathbb{N}}$ ω -associated to M of simple $k_*\hat{G}_n$ -modules M_n inductively defined by $M_0 = M$ and, denoting by φ_n the Brauer character of M_n , by $\varphi_{n+1} = \Gamma_{(\hat{G}_n, b_n)}^\omega(\varphi_n)$ for any $n \in \mathbb{N}$; explicitly, the Morita equivalence between $k_*\hat{G}_n b_n$ and $(k_*\hat{G}_n)_{\gamma_n}$ determines a simple $(k_*\hat{G}_n)_{\gamma_n}$ -module $(M_n)_{\gamma_n}$ and let us set

$$\text{End}_k(M_n)_{\gamma_n} = \text{End}_k((M_n)_{\gamma_n}) \quad 7.3.1;$$

then, choosing an embedding (cf. statement 4.4.2)

$$e_{\gamma_n} : (k_*\hat{G}_n)_{\gamma_n} \longrightarrow S_n \otimes_k k_*\hat{L}_n \quad 7.3.2,$$

the restriction *via* the embedding 4.5.2 determines a simple $k_*\hat{L}_n$ -module \dot{M}_n which becomes a simple $k_*\hat{F}_{k_*\hat{L}}(O_{\eta_n}^n)$ -module (cf. isomorphism 4.6.1); finally, we may assume that we have (cf. isomorphism 4.5.4)

$$M_{n+1} = \text{Res}_{\Phi_{S_n}^\omega(O_{\eta_n}^n) \circ \hat{F}_{\bar{e}_{\gamma_n}}(O_{\eta_n}^n)}(\dot{M}_n) \quad 7.3.3.$$

7.4. For any $n \in \mathbb{N}$, let Q^n be a *vertex* and E_n a Q^n -source of M_n ; denoting by $Q_{E_n}^n$ the corresponding local pointed group on $\text{End}_k(M_n)$, it is clear that there is a local point δ_n of Q^n on $k_*\hat{G}_n b_n$ which has a nonzero image in $(\text{End}_k(M_n))(Q_{E_n}^n)$; thus, we may assume that $P_{\gamma_n}^n$ contains $Q_{\delta_n}^n$ and it follows easily from [9 Proposition 1.6] applied to $\text{End}_k(M)$ that $Q_{\delta_n}^n$ is a radical pointed group on $k_*\hat{G}_n b_n$, so that $Q_{\delta_n}^n$ contains $O_{\eta_n}^n$ (cf. Proposition 4.2).

Lemma 7.5. *With the notation above and up to a suitable identification, the quotient $\bar{Q}^n = Q^n/O^n$ is a vertex of M_{n+1} . In particular, there is $\bar{x} \in G_{n+1}$ such that $(\bar{Q}^n)^{\bar{x}} = Q^{n+1}$.*

Proof: Since Q^n is a vertex of M_n and $P_{\gamma_n}^n$ contains $Q_{\delta_n}^n$, we have

$$\text{End}_k(M_n)_{\gamma_n}(Q^n) \neq \{0\} \quad 7.5.1$$

and therefore we still have

$$(\text{End}_k(M_{n+1}))(\bar{Q}^n) \cong (\text{End}_k(M_n))(\bar{Q}^n) \neq \{0\} \quad 7.5.2,$$

so that there is $\bar{x} \in G_{n+1}$ such that $(\bar{Q}^n)^{\bar{x}} \subset Q^{n+1}$.

Conversely, denoting by \check{Q}^{n+1} the converse image of Q^{n+1} in P^n and by x a lifting of \bar{x} to $N_{G_n}(O_{\eta_n}^n)$, we have $(Q^n)^x \subset \check{Q}^{n+1}$; moreover, since $Q^{n+1} \subset P^n$, it is clear that $S_n(\check{Q}^{n+1}) \neq \{0\}$ and therefore we get

$$\begin{aligned} (S_n \otimes_k \text{End}_k(M_n))(\check{Q}^{n+1}) \\ \cong S_n(\check{Q}^{n+1}) \otimes_k (\text{End}_k(M_{n+1}))(Q^{n+1}) \neq \{0\} \end{aligned} \quad 7.5.3$$

which implies that $\text{End}_k(M_n)_{\gamma_n}(\check{Q}^{n+1}) \neq \{0\}$ and *a fortiori* that

$$(\text{End}_k(M_n))(\check{Q}^{n+1}) \neq \{0\} \quad 7.5.4;$$

thus, \check{Q}^{n+1} is contained in a vertex of M_n and thus we have $(\bar{Q}^n)^{\bar{x}} = Q^{n+1}$.

7.6. Once again, since the sequence \mathcal{B} “stabilizes”, up to finite number of choices we may assume that $\bar{Q}^n = Q^{n+1}$ for any $n \in \mathbb{N}$; at this point, setting $Q = Q^0$, these equalities determine group homomorphisms

$$\rho_n : Q \longrightarrow Q^n \subset P^n \quad 7.6.1$$

and therefore we have a Dade Q -algebra $\text{Res}_{\rho_n}(S_n)$ for any $n \in \mathbb{N}$; then, since all but a finite number of these Dade Q -algebras are isomorphic to k , it makes sense to define the Dade Q -algebra

$$T = \bigotimes_{n \in \mathbb{N}} \text{Res}_{\rho_n}(S_n) \quad 7.6.2;$$

more generally, we denote by T_h the Dade Q^h -algebra obtained from the tensor product $\bigotimes_{n \in \mathbb{N}} \text{Res}_{\rho_{h+n}}(S_{h+n})$ for any $h \in \mathbb{N}$. We are ready to describe a vertex Q and a Q -source $E = E_0$ of M ; as it could be expected, our parameterizations agree with the correspondence exhibited by Okuyama in [8].

Proposition 7.7. *With the notation and the choice above, R is a vertex of M and, assuming that $Q = R$, an R -source E of M is determined by an R -interior algebra embedding $\text{End}_k(E) \rightarrow T$.*

Proof: We argue by induction on the “length to stabilization” of \mathcal{B} ; if this length is zero then we have $Q = \{1\} = R$ and $T \cong k$, so that everything is clear. Otherwise, considering the k^* -group \hat{G}_1 , the simple $k_*\hat{G}_1$ -module M_1 , the Fitting block sequence $\mathcal{B}_1 = \{(\hat{G}_{1+n}, b_{1+n})\}_{n \in \mathbb{N}}$ and the G_1 -conjugacy

class $(\overline{R^1}, Y^1)$ of *weights* of \hat{G}_1 determined by M_1 , it follows from the induction hypothesis that we may assume that $Q^1 = R^1$ and that an R^1 -source E_1 of M_1 is determined by an R^1 -interior algebra embedding $\text{End}_k(E_1) \rightarrow T_1$.

But, it follows from Lemma 7.5 that we may assume that Q is the converse image of Q^1 in P , and from Corollary 5.7 that R is G -conjugate to the converse image of R^1 ; consequently, R is also a vertex of M and we may assume that $R = Q$. Moreover, we clearly have a P^0 -interior algebra embedding (cf. 7.3)

$$\text{End}_k(M_0)_{\gamma_0} \longrightarrow S_0 \otimes_k \text{End}_k(M_0) \quad 7.7.2$$

and therefore, since we have $Q_{\delta_0}^0 \subset P_{\gamma_0}^0$ (cf. 7.4), we can choose an R -source E of $M_0 = M$ such that embedding 7.7.2 determines an R -interior algebra embedding

$$\text{End}_k(E) \longrightarrow \text{Res}_{\rho_0}(S_0) \otimes_k \text{End}_k(E_1) \longrightarrow \text{Res}_{\rho_0}(S_0) \otimes_k T_1 = T \quad 7.7.3.$$

We are done.

7.8. From now on, we assume that $Q^n = R^n$ for any $n \in \mathbb{N}$ and, as in Lemma 2.16 above, we denote by $R_{E_n}^n$ the corresponding local pointed group on $\text{End}_k(M_n)$; let us consider a *multiplicity* $k_* \hat{N}_{G_n}(R_{E_n}^n)$ -module V_n of M_n ; since by Proposition 7.7 we already know that $\text{End}_k(E_n)$ is a Dade R^n -algebra, it follows from Lemma 2.16 that there exists a k^* -group isomorphism

$$\hat{N}_{G_n}(R_{E_n}^n) \cong \bar{N}_{\hat{G}_n}(R_{E_n}^n) \quad 7.8.1$$

which, according to the k^* -group homomorphism 2.8.1, depends on the choice of a splitting for the k^* -group (cf. 2.9 and Proposition 7.7)

$$\hat{F}_{\text{End}_k(M_n)}(R_{E_n}^n) \cong \hat{F}_{\text{End}_k(E_n)}(R_{E_n}^n) \cong \hat{F}_{T_n}(R^n) \quad 7.8.2$$

and indeed, from our choice of the *polarization* ω , we have the splitting

$$\omega_{(R^n, T_n)} : \hat{F}_{T_n}(R^n) \longrightarrow k^* \quad 7.8.3.$$

7.9. On the other hand, for any $n \in \mathbb{N}$, let W_n be the restriction to the stabilizer $\hat{N}_{G_n}(R_{\delta_n}^n)_{E_n}$ in $\hat{N}_{G_n}(R_{\delta_n}^n)$ of the isomorphism class of E_n , of a *multiplicity* $k_* \hat{N}_{G_n}(R_{\delta_n}^n)$ -module of $R_{\delta_n}^n$ (cf. 2.5); more explicitly, denoting by $\bar{b}(\delta_n)$ the block of $\bar{C}_{\hat{G}_n}(R^n)$ determined by δ_n , since $R_{\delta_n}^n$ is a radical pointed group on $k_* \hat{G}_n$, we have (cf. 2.11.1 and 3.1)

$$k_* \bar{C}_{\hat{G}_n}(R^n) \bar{b}(\delta_n) \cong \text{End}_k(W_n) \quad 7.9.1.$$

Consequently, since $(\text{End}_k(M_n))(R_{E_n}^n) \cong \text{End}_k(V_n)$ has a $\bar{C}_{\hat{G}}(R^n)$ -interior algebra structure, it makes sense to consider $\bar{b}(\delta_n) \cdot (\text{End}_k(M_n))(R_{E_n}^n) \cdot \bar{b}(\delta_n)$ as a $\hat{N}_{G_n}(R_{E_n}^n)_{\delta_n}$ -interior algebra and, from the structural homomorphism above, we get an injective k -algebra homomorphism

$$(k_*\hat{G})(R_{\delta_n}^n) \longrightarrow \bar{b}(\delta_n) \cdot (\text{End}_k(M_n))(R_{E_n}^n) \cdot \bar{b}(\delta_n) \quad 7.9.2;$$

then, denoting by $\hat{N}_{G_n}(R_{E_n}^n)_{\delta_n}$ the stabilizer of δ_n in $\hat{N}_{G_n}(R_{E_n}^n)$ and setting

$$\hat{N}_n^{E_n} = \hat{N}_{G_n}(R_{E_n}^n)_{\delta_n} * (\hat{N}_{G_n}(R_{\delta_n}^n)_{E_n})^\circ \quad 7.9.3,$$

it follows from [9, Proposition 2.1] that, for a suitable *primitive* $\hat{N}_n^{E_n}$ -interior algebra B_n , we have a $\hat{N}_{G_n}(R_{E_n}^n)$ -interior algebra isomorphism

$$\text{End}_k(V_n) \cong \text{Ind}_{\hat{N}_{G_n}(R_{E_n}^n)_{\delta_n}}^{\hat{N}_{G_n}(R_{E_n}^n)} (k_*\bar{C}_{\hat{G}_n}(R^n)\bar{b}(\delta_n) \otimes_k B_n) \quad 7.9.4;$$

actually, it is easily checked that the subgroup

$$\bar{C}_G(R^n) \cong \bar{C}_{\hat{G}}(R^n) * \bar{C}_{\hat{G}}(R^n)^\circ \subset \hat{N}_n^{E_n} \quad 7.9.5$$

has a trivial image in B_n so that, up to an obvious identification, B_n becomes an $\hat{N}_n^{E_n}/\bar{C}_G(R^n)$ -interior algebra.

7.10. Similarly, denoting by U_n a *simple projective* $k_*\bar{N}_{\hat{G}_n}(R^n)$ -module which restricted to $N_{\hat{G}_n}(R^n)$ belongs to the isomorphism class Y^n , it follows from [16, Proposition 3.2] applied to the *primitive* $N_{\hat{G}_n}(R^n)$ -interior algebra $\text{End}_k(U_n)$ that, setting

$$\hat{N}_n = \bar{N}_{\hat{G}_n}(R_{\delta_n}^n) * \hat{N}_{G_n}(R_{\delta_n}^n)^\circ \quad 7.10.1,$$

for a suitable *primitive* \hat{N}_n -interior algebra D_n we have

$$\text{End}_k(U_n) \cong \text{Ind}_{\bar{N}_{\hat{G}_n}(R_{\delta_n}^n)}^{N_{\hat{G}_n}(R^n)} (k_*\bar{C}_{\hat{G}_n}(R^n)\bar{b}(\delta_n) \otimes_k D_n) \quad 7.10.2;$$

actually, it is clear from its very definition that D_n becomes a $\hat{N}_n/\bar{C}_{G_n}(R^n)$ -interior algebra and note that, according to homomorphism 2.8.1, we have a *canonical* k^* -group isomorphism

$$\hat{N}_n/\bar{C}_{G_n}(R^n) \cong \hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n) \quad 7.10.3.$$

In order to relate D_n with U_{n+1} , we have to consider the following k^* -group isomorphism.

Proposition 7.11. *With the notation and the choice above, there is a k^* -group isomorphism*

$$\omega\phi_n : \hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n) \cong \bar{N}_{\hat{G}_{n+1}}(R^{n+1}) \quad 7.11.1$$

such that, for any P^n -interior algebra embedding

$$e_{\gamma_n} : (k_*\hat{G}_n)_{\gamma_n} \longrightarrow S_n \otimes_k k_*\hat{L}_n \quad 7.11.2,$$

we have the commutative diagram

$$\begin{array}{ccccc} \hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n) & \cong & \hat{F}_{S_n \otimes_k k_*\hat{L}_n}(R_{S_n \times \delta_n}^n) & \cong & \hat{F}_{k_*\hat{L}_n}(R_{\delta_n}^n) \\ & & & & \wr \\ \omega\phi_n \wr & & & & \bar{N}_{\hat{L}_n}(R^n) \\ & & & & \wr \\ \bar{N}_{\hat{G}_{n+1}}(R^{n+1}) & \cong & \bar{N}_{\hat{F}_{S_n \otimes_k k_*\hat{L}_n}(O_{S_n \times \eta_n}^n)}(\bar{R}^n) & \cong & \bar{N}_{\hat{F}_{k_*\hat{L}_n}(O_{\eta_n}^n)}(\bar{R}^n) \end{array} \quad 7.11.3.$$

Proof: Choosing a P^n -interior algebra embedding $e_\gamma : (k_*\hat{G})_\gamma \rightarrow S \otimes_k k_*\hat{L}$, we have k^* -group isomorphisms (cf. 5.5.1)

$$\hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n) \xrightarrow{\hat{F}_{e_{\gamma_n}}(R_{\delta_n}^n)} \hat{F}_{S_n \otimes_k k_*\hat{L}_n}(R_{S_n \times \delta_n}^n) \xrightarrow{\Phi_{S_n}^\omega(R_{\delta_n}^n)} \hat{F}_{k_*\hat{L}_n}(R_{\delta_n}^n) \quad 7.11.4;$$

similarly, the k^* -group isomorphisms 4.5.4 applied to $O_{\eta_n}^n$ yield

$$\hat{F}_{k_*\hat{G}_n}(O_{\eta_n}^n) \xrightarrow{\hat{F}_{e_{\gamma_n}}(O_{\eta_n}^n)} \hat{F}_{S_n \otimes_k k_*\hat{L}_n}(O_{S_n \times \eta_n}^n) \xrightarrow{\Phi_{S_n}^\omega(O_{\eta_n}^n)} \hat{F}_{k_*\hat{L}_n}(O_{\eta_n}^n) \quad 7.11.5;$$

furthermore, setting $\bar{R}^n = R^n/O^n$, we have *canonical* k^* -group isomorphisms (cf. isomorphisms 4.6.2 and 5.5.2)

$$\hat{F}_{k_*\hat{L}_n}(R_{\delta_n}^n) \cong \bar{N}_{\hat{L}_n}(R^n) \cong \bar{N}_{\hat{F}_{k_*\hat{L}_n}(O_{\eta_n}^n)}(\bar{R}^n) \quad 7.11.6.$$

Now, it is clear that the commutativity of the corresponding diagram above defines a k^* -group isomorphism $\omega\phi_n$.

We claim that this k^* -group isomorphism does not depend on the choice of e_{γ_n} ; indeed, for another choice e'_{γ_n} of this embedding, it follows from Proposition 5.2 that there is $\tilde{\sigma}_n \in \text{Out}_{P^n}((k_*\hat{G}_n)_{\gamma_n})$ fulfilling

$$\tilde{e}'_{\gamma_n} = \tilde{e}_{\gamma_n} \circ \tilde{\sigma}_n = (\widetilde{\text{id}_{S_n}} \otimes \tilde{\sigma}_n) \circ \tilde{e}_{\gamma_n} \quad 7.11.7$$

and therefore, with obvious notation, we get the following commutative diagrams

$$\begin{array}{ccccc}
\hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n) & \xrightarrow{\hat{F}_{\bar{e}\gamma_n}(R_{\delta_n}^n)} & \hat{F}_{S_n \otimes_k k_* \hat{L}_n}(R_{S_n \times \delta_n}^n) & \xrightarrow{\Phi_{S_n}^\omega(R_{\delta_n}^n)} & \hat{F}_{k_* \hat{L}_n}(R_{\delta_n}^n) \\
\parallel & & \hat{F}_{\text{id}_S \otimes \hat{\sigma}}(R_{S_n \times \delta_n}^n) \wr & & \hat{F}_{\hat{\sigma}}(R_{\delta_n}^n) \wr \\
\hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n) & \xrightarrow{\hat{F}_{\bar{e}\gamma_n}(R_{\delta_n}^n)} & \hat{F}_{S_n \otimes_k k_* \hat{L}_n}(R_{S_n \times \delta_n}^n) & \xrightarrow{\Phi_{S_n}^\omega(R_{\delta_n}^n)} & \hat{F}_{k_* \hat{L}_n}(R_{\delta_n}^n)
\end{array} \quad 7.11.8$$

$$\begin{array}{ccccc}
\hat{F}_{k_*\hat{G}_n}(O_{\eta_n}^n) & \xrightarrow{\hat{F}_{\bar{e}\gamma_n}(O_{\eta_n}^n)} & \hat{F}_{S_n \otimes_k k_* \hat{L}_n}(O_{S_n \times \eta_n}^n) & \xrightarrow{\Phi_{S_n}^\omega(O_{\eta_n}^n)} & \hat{F}_{k_* \hat{L}_n}(O_{\eta_n}^n) \\
\parallel & & \hat{F}_{\text{id}_S \otimes \hat{\sigma}}(O_{S_n \times \eta_n}^n) \wr & & \hat{F}_{\hat{\sigma}}(O_{\eta_n}^n) \wr \\
\hat{F}_{k_*\hat{G}_n}(O_{\eta_n}^n) & \xrightarrow{\hat{F}_{\bar{e}\gamma_n}(O_{\eta_n}^n)} & \hat{F}_{S_n \otimes_k k_* \hat{L}_n}(O_{S_n \times \eta_n}^n) & \xrightarrow{\Phi_{S_n}^\omega(O_{\eta_n}^n)} & \hat{F}_{k_* \hat{L}_n}(O_{\eta_n}^n)
\end{array} \quad 7.11.9.$$

Now, the commutativity of the corresponding diagram above follows from these commutative diagrams and from the *naturality* of the right-hand vertical isomorphisms in the diagram. We are done.

Corollary 7.12. *With the notation and the choice above, we have an $\hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n)$ -interior algebra isomorphism*

$$D_n \cong \text{Res}_{\omega_{\phi_n}}(\text{End}_k(U_{n+1})) \quad 7.12.1.$$

Proof: Since $\text{End}_k(U_n)$ is actually isomorphic to a *block of defect zero* of the k^* -group $\bar{N}_{\hat{G}_n}(R^n)$ (cf. 7.10), it follows from [16, Theorem 3.7] and from isomorphism 7.10.2 above that D_n is isomorphic to a *block of defect zero* of the k^* -group $\hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n)$ and then isomorphism 7.12.1 easily follows from the commutativity of diagram 7.11.3 above.

7.13. On the other hand, according to Proposition 7.7, we have R^n - and R^{n+1} -interior algebra embeddings

$$\text{End}_k(E_n) \longrightarrow T_n \quad \text{and} \quad \text{End}_k(E_{n+1}) \longrightarrow T_{n+1} \quad 7.13.2;$$

but, denoting by t_n , t_{n+1} and s_n the respective *similarity* classes in the *Dade group* $\mathcal{D}_k(R^n)$ of T_n , of the restriction to R^n of T_{n+1} , and of S_n (cf. 2.13), we clearly have $t_n = s_n + t_{n+1}$ (cf. 7.6) and therefore any automorphism of R^n stabilizing s_n stabilizes t_n if and only if it stabilizes t_{n+1} ; moreover, it follows from the inclusion in 4.5.3 that $F_{k_*\hat{G}_n}(R_{\delta_n}^n)$ stabilizes s_n . Consequently, since t_n and t_{n+1} respectively determine the isomorphism classes of E_n and of the restriction to R^n of E_{n+1} , the stabilizers in $F_{k_*\hat{G}_n}(R_{\delta_n}^n)$ of these isomorphism classes coincide with each other, and therefore we have a *canonical* surjective homomorphism

$$\nu_n : \bar{N}_{G_n}(R_{\delta_n}^n)_{E_n} = \bar{N}_{G_n}(R_{E_n}^n)_{\delta_n} \longrightarrow \bar{N}_{G_{n+1}}(R_{E_{n+1}}^{n+1}) \quad 7.13.3.$$

Proposition 7.14. *With the notation and the choice above, there are a k^* -group homomorphism and a $\hat{N}_n^{E_n}$ -interior algebra isomorphism*

$$\begin{aligned} \hat{\nu}_n : \hat{N}_n^{E_n} &\longrightarrow \hat{N}_{G_{n+1}}(R_{E_{n+1}}^{n+1}) \\ f_n : B_n &\cong \text{Res}_{\nu_n} \left((\text{End}_k(M_{n+1}))(R_{E_{n+1}}^{n+1}) \right) \end{aligned} \quad 7.14.1$$

such that, for any P^n -interior algebra embedding

$$e_{\gamma_n} : (k_* \hat{G}_n)_{\gamma_n} \longrightarrow S_n \otimes_k k_* \hat{L}_n \quad 7.14.2,$$

we have the commutative diagram

$$\begin{array}{ccc} B_n & \cong & (\text{End}_k(\dot{M}_n))(R_{\dot{E}_n}^n) \\ f_n \wr \parallel & & \wr \parallel \\ (\text{End}_k(M_{n+1}))(R_{E_{n+1}}^{n+1}) & \cong & (\text{End}_k(\dot{M}_n))(\bar{R}_{\bar{E}_n}^n) \end{array} \quad 7.14.3.$$

Proof: We have a structural injective k -algebra homomorphism (cf. 7.3.1)

$$(k_* \hat{G})_{\gamma_n}(R_{\delta_n}^n) \longrightarrow (\text{End}_k(M_n))(R_{E_n}^n)_{\gamma_n} \quad 7.14.4$$

and, as in 7.9 above, denoting by C_n the centralizer in $\text{End}_k(M_n)_{\gamma_n}(R_{E_n}^n)$ of the image of $(k_* \hat{G})_{\gamma_n}(R_{\delta_n}^n)$, it follows from [9, Proposition 2.1] that we have a k -algebra isomorphism

$$\text{End}_k(M_n)_{\gamma_n}(R_{E_n}^n) \cong (k_* \hat{G})_{\gamma_n}(R^n) \otimes_k C_n \quad 7.14.5.$$

Moreover, always according to [9, Proposition 2.1], the obvious commutative diagram

$$\begin{array}{ccc} (k_* \hat{G})_{\gamma_n}(R_{\delta_n}^n) & \longrightarrow & (k_* \hat{G})(R_{\delta_n}^n) \\ \downarrow & & \downarrow \\ \text{End}_k(M_n)_{\gamma_n}(R_{E_n}^n) & \longrightarrow & \bar{b}(\delta_n) \cdot (\text{End}_k(M_n))(R_{E_n}^n) \cdot \bar{b}(\delta_n) \end{array} \quad 7.14.6$$

induce a *canonical* k -algebra isomorphism $C_n \cong B_n$ which allows us to identify both centralizers.

Choosing a P^n -interior algebra embedding (cf. statement 4.4.2)

$$e_{\gamma_n} : (k_* \hat{G}_n)_{\gamma_n} \longrightarrow S_n \otimes_k k_* \hat{L}_n \quad 7.14.7,$$

note that $(k_* \hat{L}_n)(R_{\delta_n}^n) \cong k$ since $R_{\delta_n}^n$ and $R_{\dot{\delta}_n}^n$ are *radical* and therefore they are *selfcentralizing*; then, the corresponding commutative diagram

$$\begin{array}{ccc} (k_* \hat{G}_n)_{\gamma_n}(R_{\delta_n}^n) & \longrightarrow & S_n(R^n) \\ \downarrow & & \downarrow \\ (\text{End}_k(M_n))_{\gamma_n}(R_{E_n}^n) & \longrightarrow & S_n(R^n) \otimes_k (\text{End}_k(\dot{M}_n))(R_{E_n}^n) \end{array} \quad 7.14.8$$

and the argument above yield the top k -algebra isomorphism

$$B_n \cong (\text{End}_k(\dot{M}_n))(R_{\dot{E}_n}^n) \quad 7.14.9$$

in the diagram 7.14.3 above, and the k^* -group isomorphism

$$\hat{N}_n^{E_n}/C_G(R^n) \cong \hat{N}_{L_n}(R_{\dot{E}_n}^n) \quad 7.14.10.$$

Moreover, according to 7.3, we get

$$\text{End}_k(\dot{M}_n) \cong \text{End}_k(\dot{M}_n) \cong \text{End}_k(M_{n+1}) \quad 7.14.11$$

with the interior structures coming from the k^* -group isomorphisms (cf. 4.5.4 and 4.6.1)

$$\hat{L}_n/O^n \cong \hat{F}_{k_*\hat{L}_n}(O_{\eta_n}^n) \cong \hat{G}_{n+1} \quad 7.14.12;$$

consequently, we still get

$$\begin{aligned} (\text{End}_k(\dot{M}_n))(R_{\dot{E}_n}^n) &\cong (\text{End}_k(\dot{M}_n))(R_{\dot{E}_n}^n) \\ &\cong (\text{End}_k(M_{n+1}))(R_{E_{n+1}}^{n+1}) \end{aligned} \quad 7.14.13$$

with the interior structures coming from the k -group isomorphisms

$$\hat{N}_{L_n}(R_{\dot{E}_n}^n) \cong \hat{N}_{F_{k_*\hat{L}_n}(O_{\eta_n}^n)}(\bar{R}_{\dot{E}_n}^n) \cong \hat{N}_{G_{n+1}}(R_{E_{n+1}}^{n+1}) \quad 7.14.14.$$

Finally, for a particular choice of e_{γ_n} , the commutativity of diagram 7.14.3 induce the isomorphism f_n , and the k^* -group isomorphisms 7.14.10 and 7.14.14 determine the k^* -group homomorphism $\hat{\nu}_n$. Once again f_n and $\hat{\nu}_n$ do not depend on the choice of e_{γ_n} ; indeed, for another choice e'_{γ_n} of this embedding, it follows from Proposition 5.2 that there is $\tilde{\sigma}_n \in \text{Out}_{P^n}((k_*\hat{G}_n)_{\gamma_n})$ fulfilling

$$\tilde{e}'_{\gamma_n} = \tilde{e}_{\gamma_n} \circ \tilde{\sigma}_n = (\tilde{\text{id}}_{S_n} \otimes \tilde{\sigma}_n) \circ \tilde{e}_{\gamma_n} \quad 7.14.15$$

and we get commutative diagrams as above.

7.15. We are ready to describe the *multiplicity* $\hat{N}_G(R_E)$ -module of M ; first of all, from the very definition of $N_G(\mathcal{B})$ (cf. 6.2), we get a sequence of k^* -groups $\hat{N}_G^n(\mathcal{B})$, with the same k^* -quotient $N_G(\mathcal{B})$, and of k^* -group homomorphisms $\hat{\mu}_n : \hat{N}_G^n(\mathcal{B}) \rightarrow \hat{G}_n$ inductively defined as follows; the k^* -group $\hat{N}_G^0(\mathcal{B})$ is just the converse image of $N_G(\mathcal{B})$ in \hat{G} and $\hat{\mu}_0$ the inclusion map; then, for any $n \geq 1$, arguing by induction on n it is easily checked that the image of $\hat{\mu}_{n-1}$ normalizes the pointed group $O_{\eta_{n-1}}^{n-1}$ on $k_*\hat{G}_{n-1}$ and therefore $\hat{\mu}_{n-1}$ induces a group homomorphism μ_{n-1} from $N_G(\mathcal{B})$ to $N_{G_{n-1}}(O_{\eta_{n-1}}^{n-1})$;

since we have (cf. statement 2.11.2)

$$N_{G_{n-1}}(O_{\eta_{n-1}}^{n-1})/O^{n-1}.C_{G_{n-1}}(O^{n-1}) \cong F_{k_*\hat{G}_{n-1}}(O_{\eta_{n-1}}^{n-1}) \quad 7.15.1,$$

we define $\hat{N}_G^n(\mathcal{B})$ and $\hat{\mu}_n$ by the following *pull-back*

$$\begin{array}{ccc} N_G(\mathcal{B}) & \longrightarrow & F_{k_*\hat{G}_{n-1}}(O_{\eta_{n-1}}^{n-1}) \\ \uparrow & & \uparrow \\ \hat{N}_G^n(\mathcal{B}) & \xrightarrow{\hat{\mu}_n} & \hat{G}_n \end{array} \quad 7.15.2.$$

7.16. Now, consider the *pointed vertex sequence* $\mathcal{R} = \{R_{\delta_n}^n\}_{n \in \mathbb{N}}$ of M associated to ω (cf. 7.4 and 7.8), and denote by $N_G(\mathcal{R})$ the stabilizer of \mathcal{R} in G ; clearly, $N_G(\mathcal{B})$ contains $N_G(\mathcal{R})$ and we denote by $\hat{N}_G^n(\mathcal{R})$ the corresponding k^* -subgroup of $\hat{N}_G^n(\mathcal{B})$. Moreover, arguing by induction on n , it is easily checked that the subgroup $\mu_n(N_G(\mathcal{R})) \subset G_n$ normalizes the pointed group $R_{\delta_n}^n$ on $k_*\hat{G}_n$; thus, setting $\bar{N}_G(\mathcal{R}) = N_G(\mathcal{R})/R$, we get a k^* -group $\hat{N}_G^{\delta_n}(\mathcal{R})$ from the following *pull-back*

$$\begin{array}{ccc} \bar{N}_G(\mathcal{R}) & \xrightarrow{\bar{\mu}_n} & \bar{N}_{G_n}(R_{\delta_n}^n) \\ \uparrow & & \uparrow \\ \hat{N}_G^{\delta_n}(\mathcal{R}) & \xrightarrow{\hat{\mu}_n} & \hat{N}_{G_n}(R_{\delta_n}^n) \end{array} \quad 7.16.1$$

and then we have the $k_*\hat{N}_G^{\delta_n}(\mathcal{R})$ -module $\text{Res}_{\hat{\mu}_n^{\delta_n}}(W_n)$ for any $n \in \mathbb{N}$ (cf. 7.9.1).

7.17. On the other hand, from the k^* -group homomorphism 2.8.1 and from Proposition 7.11, for any $n \in \mathbb{N}$ we get the k^* -group homomorphisms

$$\begin{array}{ccc} \bar{N}_{\hat{G}_n}(R_{\delta_n}^n) \star \hat{N}_{G_n}(R_{\delta_n}^n)^\circ & \longrightarrow & \hat{F}_{k_*\hat{G}_n}(R_{\delta_n}^n) \xrightarrow{\omega_{\phi_n}} \bar{N}_{\hat{G}_{n+1}}(R^{n+1}) \\ \hat{\mu}_n \star \hat{\mu}_n^{\delta_n} \uparrow & & \hat{\mu}_{n+1} \uparrow \\ \hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{\delta_n}(\mathcal{R})^\circ & & \hat{N}_G^{n+1}(\mathcal{R}) \end{array} \quad 7.17.1$$

and therefore, since all the bottom k^* -groups admit the same k^* -quotient, we still get a k^* -group isomorphism

$$\omega\Psi_n : \hat{N}_G^{\delta_n}(\mathcal{R}) \cong \hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{n+1}(\mathcal{R})^\circ \quad 7.17.2.$$

But, for n big enough we have

$$W_n \cong k \quad \text{and} \quad \hat{N}_G^n(\mathcal{B}) \cong k^* \times N_G(\mathcal{B}) \quad 7.17.3;$$

moreover, note that $\hat{N}_G^0(\mathcal{B})$ coincides with the converse image $N_{\hat{G}}(\mathcal{B})$ of $N_G(\mathcal{B})$ in \hat{G} and similarly we set $N_{\hat{G}}(\mathcal{R}) = \hat{N}_G^0(\mathcal{R})$ and $\bar{N}_{\hat{G}}(\mathcal{R}) = N_{\hat{G}}(\mathcal{R})/R$;

in particular, the following tensor product

$${}^\omega W = \bigotimes_{n \in \mathbb{N}} \text{Res}_{\hat{\mu}_n^{\delta_n} \circ (\omega \Psi_n)^{-1}}(W_n) \quad 7.17.4$$

makes sense and it is clearly a $k_* \bar{N}_G(\mathcal{R})$ -module.

7.18. Finally, it follows from 4.5.3 that $F_{k_* \hat{G}_n}(R_{\delta_n}^n)$ stabilizes the isomorphism class of $\text{Res}_{R^n}^{P^n}(S_n)$ and therefore $\bar{N}_G(\mathcal{R})$ stabilizes the *similarity* class of T_n and, in particular, the isomorphism class of E_n for any $n \in \mathbb{N}$ (cf. 7.6 and Proposition 7.7); hence, we get again a k^* -group $\hat{N}_G^{E_n}(\mathcal{R})$ from the following *pull-back*

$$\begin{array}{ccc} \bar{N}_G(\mathcal{R}) & \xrightarrow{\bar{\mu}_n} & \bar{N}_{G_n}(R_{\delta_n}^n)_{E_n} \\ \uparrow & & \uparrow \\ \hat{N}_G^{E_n}(\mathcal{R}) & \xrightarrow{\hat{\mu}_n^{E_n}} & \hat{N}_{G_n}(R_{E_n}^n)_{\delta_n} \end{array} \quad 7.18.1$$

Similarly, from Proposition 7.14, for any $n \in \mathbb{N}$ we get the k^* -group homomorphisms

$$\begin{array}{ccc} \hat{N}_{G_n}(R_{E_n}^n)_{\delta_n} \star (\hat{N}_{G_n}(R_{\delta_n}^n)_{E_n})^\circ & \xrightarrow{\hat{\nu}_n} & \hat{N}_{G_{n+1}}(R_{E_{n+1}}^{n+1}) \\ \hat{\mu}_n^{E_n} \star \hat{\mu}_n^{\delta_n} \uparrow & & \hat{\mu}_{n+1}^{E_{n+1}} \uparrow \\ \hat{N}_G^{E_n}(\mathcal{R}) \star \hat{N}_G^{\delta_n}(\mathcal{R})^\circ & & \hat{N}_G^{E_{n+1}}(\mathcal{R}) \end{array} \quad 7.18.2$$

and therefore, since all the bottom k^* -groups admit the same k^* -quotient, we still get a k^* -group isomorphism

$$\Psi_n : \hat{N}_G^{\delta_n}(\mathcal{R}) \cong \hat{N}_G^{E_n}(\mathcal{R}) \star \hat{N}_G^{E_{n+1}}(\mathcal{R})^\circ \quad 7.18.3;$$

thus, the following tensor product

$$W = \bigotimes_{n \in \mathbb{N}} \text{Res}_{\hat{\mu}_n^{\delta_n} \circ (\Psi_n)^{-1}}(W_n) \quad 7.18.4$$

makes sense and it is clearly a $k_* \hat{N}_G^{E_0}(\mathcal{R})$ -module. As above, we set $R = R^0$, $E = E_0$, $V = V_0$ and $U = U_0$.

Theorem 7.19. *With the notation and the choice above, we have natural $k_* \bar{N}_G(R)$ - and $k_* \hat{N}_G(R_E)$ -module isomorphisms*

$$U \cong \text{Ind}_{\bar{N}_G(\mathcal{R})}^{\bar{N}_G(R)}({}^\omega W) \quad \text{and} \quad V \cong \text{Ind}_{\hat{N}_G^E(\mathcal{R})}^{\hat{N}_G(R_E)}(W) \quad 7.19.1.$$

Proof: Once again, we can argue by induction on the “length to stabilization” of \mathcal{B} . If this length is zero then the block b_0 is already of *defect zero* and therefore everything is trivial so that isomorphisms 7.19.1 above are trivially true.

If the “length to stabilization” is not zero then we consider the *Fitting block sequence* $\mathcal{B}_1 = \{(\hat{G}_{1+n}, b_{1+n})\}_{n \in \mathbb{N}}$ of \hat{G}_1 and the corresponding *weight sequence* $\{(\bar{R}^{1+n}, Y^{1+n})\}_{n \in \mathbb{N}}$ and *simple sequence* $\{M_{1+n}\}_{n \in \mathbb{N}}$ ω -associated to \mathcal{B}_1 ; *mutatis mutandis*, we consider the corresponding *pointed vertex sequence* $\mathcal{R}_1 = \{R_{\delta_{n+1}}^{n+1}\}_{n \in \mathbb{N}}$ of M_1 associated to ω , and denote by $N_{G_1}(\mathcal{R}_1)$ and $N_{\hat{G}_1}(\mathcal{R}_1)$ the respective stabilizers of \mathcal{R}_1 in G_1 and in \hat{G}_1 .

Moreover, from the corresponding *pull-back* 7.16.1, for any $n \geq 1$ we get a k^* -group $\hat{N}_{G_1}^{\delta_n}(\mathcal{R}_1)$ of k^* -quotient $\bar{N}_{G_1}(\mathcal{R}_1)$ and a k^* -group homomorphism

$$\hat{N}_{G_1}^{\delta_n}(\mathcal{R}_1) \xrightarrow{\hat{\mu}_{1,n}^{\delta_n}} \hat{N}_{G_n}(R_{\delta_n}^n) \quad 7.19.2,$$

so that we still get the $k_*\hat{N}_{G_1}^{\delta_n}(\mathcal{R}_1)$ -module $\text{Res}_{\hat{\mu}_{1,n}^{\delta_n}}(W_n)$. Analogously, for any $n \geq 1$ we still get the corresponding k^* -group isomorphisms 7.17.2 and 7.18.3

$$\begin{aligned} \omega\Psi_{1,n} : \hat{N}_{G_1}^{\delta_n}(\mathcal{R}_1) &\cong \hat{N}_{G_1}^n(\mathcal{R}_1) \star \hat{N}_{G_1}^{n+1}(\mathcal{R}_1)^\circ \\ \Psi_{1,n} : \hat{N}_{G_1}^{\delta_n}(\mathcal{R}_1) &\cong \hat{N}_{G_1}^{E_n}(\mathcal{R}_1) \star \hat{N}_{G_1}^{E_{n+1}}(\mathcal{R}_1)^\circ \end{aligned} \quad 7.19.3,$$

and, once again, the following tensor products

$$\begin{aligned} {}^\omega W^1 &= \bigotimes_{n \geq 1} \text{Res}_{\hat{\mu}_{1,n}^{\delta_n} \circ (\omega\Psi_{1,n})^{-1}}(W_n) \\ W^1 &= \bigotimes_{n \geq 1} \text{Res}_{\hat{\mu}_{1,n}^{\delta_n} \circ (\Psi_{1,n})^{-1}}(W_n) \end{aligned} \quad 7.19.4$$

make sense and respectively become $k_*\bar{N}_{\hat{G}_1}(\mathcal{R}_1)$ - and $k_*\hat{N}_{G_1}^{E_1}(\mathcal{R}_1)$ -modules.

At this point, it follows from the induction hypothesis that we have *natural* $k_*\bar{N}_{\hat{G}_1}(R^1)$ - and $k_*\bar{N}_{\hat{G}_1}(R_{E_1}^1)$ -module isomorphisms

$$U_1 \cong \text{Ind}_{\bar{N}_{\hat{G}_1}(\mathcal{R}_1)}^{\bar{N}_{\hat{G}_1}(R^1)}({}^\omega W^1) \quad \text{and} \quad V_1 \cong \text{Ind}_{\hat{N}_{G_1}^{E_1}(\mathcal{R}_1)}^{\hat{N}_{G_1}(R_{E_1}^1)}(W^1) \quad 7.19.5.$$

But, it follows from isomorphisms 7.9.4 and 7.10.2, and from Corollary 7.12 and Proposition 7.14 that, considering the surjective k^* -group homomorphism (cf. 2.8.1 and Propositions 7.11 and 7.14)

$$\begin{aligned} \hat{N}_0 &= \bar{N}_{\hat{G}}(R_\delta) \star \hat{N}_{\hat{G}}(R_\delta)^\circ \longrightarrow \hat{F}_{k_*\hat{G}}(R_\delta) \xrightarrow{\omega\phi_0} \bar{N}_{\hat{G}_1}(R^1) \\ \hat{N}_0^{E_1} &= \hat{N}_{\hat{G}}(R_E)_\delta \star (\hat{N}_{\hat{G}}(R_\delta)_E)^\circ \longrightarrow \hat{N}_{G_1}(R_{E_1}^1) \end{aligned} \quad 7.19.6$$

and denoting by \check{U}_1 and \check{V}_1 the corresponding restrictions of U_1 to \hat{N}_0 and of V_1 to \hat{N}_0^E , we have

$$U \cong \text{Ind}_{\bar{N}_G(R_\delta)}^{\bar{N}_G(R)}(W_0 \otimes_k \check{U}_1) \quad \text{and} \quad V \cong \text{Ind}_{\bar{N}_G(R_E)_\delta}^{\bar{N}_G(R_E)}(W_0 \otimes_k \check{V}_1) \quad 7.19.7.$$

Furthermore, it is easily checked that the image of $\bar{N}_G(\mathcal{R}) \subset \bar{N}_G(R_\delta)$ in $\bar{N}_{G_1}(R^1)$ throughout the k^* -quotient of homomorphism 7.19.2 is contained in $\bar{N}_{G_1}(\mathcal{R}_1)$ and then that this k^* -group homomorphism induces k^* -group homomorphisms (cf. 7.17.1 and 7.18.2)

$$\begin{aligned} \bar{N}_G(\mathcal{R}) \star \hat{N}_G^\delta(\mathcal{R})^\circ &\cong \hat{N}_G^1(\mathcal{R}) \longrightarrow \bar{N}_{G_1}(\mathcal{R}_1) \\ \hat{N}_G^E(\mathcal{R}) \star \hat{N}_G^\delta(\mathcal{R})^\circ &\cong \hat{N}_G^{E_1}(\mathcal{R}) \longrightarrow \hat{N}_{G_1}^{E_1}(\mathcal{R}_1) \end{aligned} \quad 7.19.8;$$

thus, denoting by ${}^\omega \check{W}^1$ and by \check{W}^1 the corresponding restrictions of ${}^\omega W^1$ to $\hat{N}_G^1(\mathcal{R})$ and of W^1 to $\hat{N}_G^{E_1}(\mathcal{R})$ we have (cf. 7.19.5)

$$\check{U}_1 = \text{Ind}_{\hat{N}_G^1(\mathcal{R})}^{\hat{N}_0}({}^\omega \check{W}^1) \quad \text{and} \quad \check{V}_1 = \text{Ind}_{\hat{N}_G^{E_1}(\mathcal{R})}^{\hat{N}_0^E}(\check{W}^1) \quad 7.19.9.$$

More explicitly, for any $n \geq 1$ the following diagrams of k^* -group homomorphisms

$$\begin{array}{ccccc} \hat{N}_{G_1}^n(\mathcal{R}_1) \star \hat{N}_{G_1}^{n+1}(\mathcal{R}_1)^\circ & \xrightarrow{({}^\omega \Psi_{1,n})^{-1}} & \hat{N}_{G_1}^{\delta n}(\mathcal{R}_1) & \xrightarrow{\hat{\mu}_{1,n}^{\delta n}} & \hat{N}_{G_n}(R_{\delta_n}^n) \\ \uparrow & & \uparrow & & \parallel \\ \hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{n+1}(\mathcal{R})^\circ & \xrightarrow{({}^\omega \Psi_n)^{-1}} & \hat{N}_G^{\delta n}(\mathcal{R}) & \xrightarrow{\hat{\mu}_n^{\delta n}} & \hat{N}_{G_n}(R_{\delta_n}^n) \end{array} \quad 7.19.10$$

$$\begin{array}{ccccc} \hat{N}_{G_1}^{E_n}(\mathcal{R}_1) \star \hat{N}_{G_1}^{E_{n+1}}(\mathcal{R}_1)^\circ & \xrightarrow{(\Psi_{1,n})^{-1}} & \hat{N}_{G_1}^{\delta n}(\mathcal{R}_1) & \xrightarrow{\hat{\mu}_{1,n}^{\delta n}} & \hat{N}_{G_n}(R_{\delta_n}^n) \\ \uparrow & & \uparrow & & \parallel \\ \hat{N}_G^{E_n}(\mathcal{R}) \star \hat{N}_G^{E_{n+1}}(\mathcal{R})^\circ & \xrightarrow{(\Psi_n)^{-1}} & \hat{N}_G^{\delta n}(\mathcal{R}) & \xrightarrow{\hat{\mu}_n^{\delta n}} & \hat{N}_{G_n}(R_{\delta_n}^n) \end{array} \quad 7.19.11$$

are commutative since all the vertical arrows are defined by *pull-back via* the group homomorphism $N_G(\mathcal{R}) \rightarrow N_{G_1}(\mathcal{R}_1)$ determined by the k^* -quotient of homomorphism 7.19.2; hence, we actually get a $k_* \hat{N}_G^1(\mathcal{R})$ - and a $k_* \hat{N}_G^{E_1}(\mathcal{R})$ -module isomorphisms

$$\begin{aligned} {}^\omega \check{W}^1 &\cong \bigotimes_{n \geq 1} \text{Res}_{\hat{\mu}_n^{\delta n} \circ ({}^\omega \Psi_n)^{-1}}(W_n) \\ \check{W}^1 &\cong \bigotimes_{n \geq 1} \text{Res}_{\hat{\mu}_n^{\delta n} \circ (\Psi_n)^{-1}}(W_n) \end{aligned} \quad 7.19.12$$

Consequently, from the *Frobenius property*, we get a $k_*\tilde{N}_{\hat{G}}(R_\delta)$ -module isomorphism

$$\begin{aligned} W_0 \otimes_k \check{U}_1 &\cong \text{Ind}_{\tilde{N}_{\hat{G}}(\mathcal{R})}^{\tilde{N}_{\hat{G}}(R_\delta)} (\text{Res}_{\hat{\mu}_0^\delta \circ (\omega\Psi_0)^{-1}}(W_0) \otimes_k {}^\omega \check{W}^1) \\ &\cong \text{Ind}_{\tilde{N}_{\hat{G}}(\mathcal{R})}^{\tilde{N}_{\hat{G}}(R_\delta)} (W) \end{aligned} \quad 7.19.13$$

and therefore from the left-hand isomorphism in 7.19.7 we obtain the left-hand isomorphism in 7.19.1. Similarly, we get a $k_*\tilde{N}_{\hat{G}}(R_\delta)_E$ -module isomorphism

$$\begin{aligned} W_0 \otimes_k \check{V}_1 &\cong \text{Ind}_{\hat{N}_G^E(\mathcal{R})}^{\hat{N}_G^E(R_E)_\delta} (\text{Res}_{\hat{\mu}_0^\delta \circ (\Psi_0)^{-1}}(W_0) \otimes_k \check{W}^1) \\ &\cong \text{Ind}_{\hat{N}_G^E(\mathcal{R})}^{\tilde{N}_{\hat{G}}(R_\delta)_E} (W) \end{aligned} \quad 7.19.14$$

and therefore from the right-hand isomorphism in 7.19.7 we obtain the right-hand isomorphism in 7.19.1. We are done.

7.20. In order to compare both isomorphisms in 7.19.1, note that from homomorphism 2.8.2 and from our choice of a *polarization* ω we have a k^* -group homomorphism

$$\hat{N}_G(R_E)^\circ \star \tilde{N}_{\hat{G}}(R_E) \longrightarrow \hat{F}_T(R) \xrightarrow{\omega_{(R,T)}} k^* \quad 7.20.1$$

which determines a k^* -group isomorphism $\tilde{N}_{\hat{G}}(R_E) \cong \hat{N}_G(R_E)$; let us denote by ${}^\omega V$ the restriction of V throughout this isomorphism. Similarly, for any $n \in \mathbb{N}$, the $k_* (\hat{N}_G^{E_n}(\mathcal{R}) \star \hat{N}_G^{E_{n+1}}(\mathcal{R})^\circ)$ -module $\text{Res}_{\hat{\mu}_n^\delta \circ (\Psi_n)^{-1}}(W_n)$ restricted throughout the composed k^* -group isomorphism

$$\hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{n+1}(\mathcal{R})^\circ \xrightarrow{({}^\omega \Psi_n)^{-1}} \hat{N}_G^{\delta_n}(\mathcal{R}) \xrightarrow{\Psi_n} \hat{N}_G^{E_n}(\mathcal{R}) \star \hat{N}_G^{E_{n+1}}(\mathcal{R})^\circ \quad 7.20.2$$

coincides with ${}^\omega W_n$.

7.21. But, according to the right-hand k^* -group isomorphism in 4.5.4, the corresponding splitting

$$(\hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{n+1}(\mathcal{R})^\circ) \star (\hat{N}_G^{E_n}(\mathcal{R}) \star \hat{N}_G^{E_{n+1}}(\mathcal{R})^\circ)^\circ \longrightarrow k^* \quad 7.21.1$$

comes from $\omega_{(R^n, \text{Res}_{R^n}^{P^n}(S_n))} : \hat{F}_{S_n}(R^n) \rightarrow k^*$ and needs not coincide with the splitting

$$(\hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{E_n}(\mathcal{R})^\circ) \star (\hat{N}_G^{n+1}(\mathcal{R}) \star \hat{N}_G^{E_{n+1}}(\mathcal{R})^\circ)^\circ \longrightarrow k^* \quad 7.21.2$$

coming from (cf. 2.8.1)

$$\omega_{(R^n, T_n)} : \hat{F}_{T_n}(R^n) \rightarrow k^* \quad \text{and} \quad \omega_{(R^{n+1}, T_{n+1})} : \hat{F}_{T_{n+1}}(R^{n+1}) \rightarrow k^* \quad 7.21.3;$$

that is to say, this splitting determines a new k^* -group isomorphism between $\hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{n+1}(\mathcal{R})^\circ$ and $\hat{N}_G^{E_n}(\mathcal{R}) \star \hat{N}_G^{E_{n+1}}(\mathcal{R})^\circ$; thus, this isomorphism and $\psi_n \circ (\omega \psi_n)^{-1}$ determine an automorphism $\omega \theta_n$ of $\hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{n+1}(\mathcal{R})^\circ$. Then, it is clear that the product of all these automorphisms defines an automorphism $\omega \theta$ of $\hat{N}_G^0(\mathcal{R}) = \tilde{N}_G(\mathcal{R})$ and that the right-hand isomorphism in 7.19.1 implies the following result.

Corollary 7.22. *With the notation and the choice above, we have a natural $k_* \tilde{N}_G(R_E)$ -module isomorphism*

$$\omega V \cong \text{Ind}_{\tilde{N}_G(\mathcal{R})}^{\tilde{N}_G(R_E)} (\text{Res}_{\omega \theta}(\omega W)) \quad 7.22.1.$$

Appendix: The odd order case

A.1. Assume that $p \neq 2$ and let \hat{G} be a k^* -group with finite k^* -quotient G of *odd* order. In this case, by the fundamental Feit-Thompson Theorem [3], G is *solvable* and therefore, for any choice of a *polarization* ω , Theorem 6.5 above supplies a *natural* bijection

$$\text{Irr}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G}) \quad A.1.1;$$

actually, it suffices to consider ω over the *torsion* subcategory $\mathfrak{D}_k^{\text{tor}}$ (cf. Remark 4.7); further, the oddness of our groups only demands the choice of a splitting for the k^* -subgroup $\mathbb{O}^2(\hat{F}_S(P))$ of $\hat{F}_S(P)$ for any $\mathfrak{D}_k^{\text{tor}}$ -object (P, S) .

A.2. That is to say, in the present situation we can replace \mathfrak{D}_k , $\hat{\mathfrak{f}}$ and ω (cf. 2.5) by the *full* subcategory $\mathfrak{D}_k^{\text{tor}}$ of \mathfrak{D}_k , by the subfunctor of $\hat{\mathfrak{f}}$

$${}^2\hat{\mathfrak{f}} : \mathfrak{D}_k^{\text{tor}} \longrightarrow k^* \text{-}\mathfrak{Gr} \quad A.1.2$$

mapping any $\mathfrak{D}_k^{\text{tor}}$ -object (P, S) on the k^* -group $\mathbb{O}^2(\hat{F}_S(P))$, and finally by a *natural* map ${}^2\omega : {}^2\hat{\mathfrak{f}} \longrightarrow k^*$ fulfilling the condition in 2.15.1 — called a *odd-polarization*. Although any *odd-polarization* can be easily extended to a *polarization*, the point is that there is a unique *odd-polarization* compatible with the *tensor product* of Dade P -algebras. We borrow the notation from [15, Chap. 9] and denote by $\mathcal{D}_k^{\text{tor}}(P)$ the subgroup of *torsion* elements of $\mathcal{D}_k(P)$; actually, it is known that all the nontrivial *torsion* elements of $\mathcal{D}_k(P)$ have order 2 [15, 8.16 and Corollary 8.22] or, equivalently, that $S \cong S^\circ$ for any $\mathfrak{D}_k^{\text{tor}}$ -object (P, S) .

Theorem A.3. *There is a unique odd-polarization ${}^2\omega$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{O}^2(\hat{F}_S(P)) \hat{\cap} \mathbb{O}^2(\hat{F}_{S'}(P)) & \xrightarrow{\hat{\nu}_{P,S,S'}} & \mathbb{O}^2(\hat{F}_{S \otimes_k S'}(P)) \\ {}^2\omega_{(P,S)} \hat{\times} {}^2\omega_{(P,S')} & \searrow & \swarrow {}^2\omega_{(P,S \otimes_k S')} \\ & k^* & \end{array} \quad \text{A.3.1}$$

for any pair of $\mathfrak{D}_k^{\text{tor}}$ -objects (P, S) and (P, S') . Moreover, for any normal subgroup Q of P , setting $T = \text{Res}_Q^P(S)$ and $\bar{P} = P/Q$, and denoting by $\hat{F}_S(P)_Q$ the stabilizer of Q in $\hat{F}_S(P)$, the following diagram is also commutative

$$\begin{array}{ccc} \mathbb{O}^2(\hat{F}_S(P)_Q) & \xrightarrow{\Delta_{P,S,Q}} & \mathbb{O}^2(\hat{F}_T(Q)) \hat{\times} \mathbb{O}^2(\hat{F}_{S(Q)}(\bar{P})) \\ {}^2\omega_{(P,S)} & \searrow & \swarrow {}^2\omega_{(Q,T)} \hat{\times} {}^2\omega_{(\bar{P},S(Q))} \\ & k^* & \end{array} \quad \text{A.3.2}$$

Proof: Let ω be a polarization [15, Theorem 9.21]; for any $\mathfrak{D}_k^{\text{tor}}$ -object (P, S) , it is clear that there is a group homomorphism $\beta_{(P,S)} : \mathbb{O}^2(F_S(P)) \rightarrow k^*$ fulfilling

$$(\omega_{(P,S)} \hat{\times} \omega_{(P,S)})(\hat{\varphi} \cdot \hat{\varphi}) = \beta_{(P,S)}(\varphi) \omega_{(P,S \otimes_k S)}(\hat{\nu}_{P,S,S'}(\hat{\varphi} \cdot \hat{\varphi})) \quad \text{A.3.3}$$

for any $\hat{\varphi} \in \mathbb{O}^2(\hat{F}_S(P))$, where $\hat{\varphi} \cdot \hat{\varphi}$ denotes the image of $(\hat{\varphi}, \hat{\varphi})$ in the k^* -group (cf. 2.2)

$$\hat{F}_S(P) \hat{\cap} \hat{F}_S(P) = \hat{F}_S(P) \star \hat{F}_S(P) \quad \text{A.3.4}$$

and φ is the image of $\hat{\varphi}$ in $\mathbb{O}^2(F_S(P))$; then, there is a *unique* group homomorphism $\alpha_{(P,S)} : \mathbb{O}^2(F_S(P)) \rightarrow k^*$ fulfilling $(\alpha_{(P,S)})^2 = \beta_{(P,S)}$ and we claim that it suffices to define

$${}^2\omega_{(P,S)}(\hat{\varphi}) = \alpha_{(P,S)}(\varphi)^{-1} \omega_{(P,S)}(\hat{\varphi}) \quad \text{A.3.5}$$

for any $\hat{\varphi} \in \mathbb{O}^2(\hat{F}_S(P))$.

In any case, note that the uniqueness of ${}^2\omega_{(P,S)}$ follows from the uniqueness of $\alpha_{(P,S)}$. The commutativity of diagram A.3.1 for $S' = S$ follows from our very definition; otherwise, the diagrams corresponding to the pairs of Dade P -algebras $(S \otimes_k S', S \otimes_k S')$, (S, S) and (S', S') are certainly commutative and then the commutativity of diagram A.3.1 follows.

Moreover, once again it is clear that there is a group homomorphism

$$\gamma_{(P,S,Q)} : \mathbb{O}^2(F_S(P)_Q) \longrightarrow k^* \quad \text{A.3.6}$$

such that, for any $\hat{\varphi} \in \mathbb{O}^2(\hat{F}_S(P)_Q)$, we have

$${}^2\omega_{(P,S)}(\hat{\varphi}) = \gamma_{(P,S,Q)}(\varphi) ({}^2\omega_{(Q,T)} \hat{\times} {}^2\omega_{(\bar{P},S(Q))})(\Delta_{P,S,Q}(\hat{\varphi})) \quad \text{A.3.7}$$

But, it follows from [15, Proposition 9.16] that the diagram

$$\begin{array}{ccc} \hat{F}_S(P)_Q \star \hat{F}_S(P)_Q & \longrightarrow & (\hat{F}_T(Q) \star \hat{F}_T(Q)) \hat{\times} (\hat{F}_{S(Q)}(\bar{P}) \star \hat{F}_{S(Q)}(\bar{P})) \\ \downarrow \hat{\nu}_{P,S,S} & & \downarrow \hat{\nu}_{Q,T,T} \hat{\times} \hat{\nu}_{\bar{P},S(Q),S(Q)} \\ \hat{F}_{S \otimes_k S}(P)_Q & \longrightarrow & \hat{F}_{T \otimes_k T}(Q) \hat{\times} \hat{F}_{(S \otimes_k S)(Q)}(\bar{P}) \end{array} \quad A.3.8$$

is commutative; moreover, since the Dade P -algebra

$$S \otimes_k S \cong S \otimes_k S^\circ \cong \text{End}_k(S) \quad A.3.9$$

is *similar* to k , the corresponding diagram A.3.2 is clearly commutative. Consequently, for any $\hat{\varphi} \in \mathbb{O}^2(\hat{F}_S(P)_Q)$, the element $({}^2\omega_{(P,S)} \hat{\times} {}^2\omega_{(P,S)})(\hat{\varphi} \cdot \hat{\varphi})$ coincides with the image of $\Delta_{P,S,Q}(\hat{\varphi}) \cdot \Delta_{P,S,Q}(\hat{\varphi})$ throughout the map

$$({}^2\omega_{(Q,T)} \hat{\times} {}^2\omega_{(Q,T)}) \hat{\times} ({}^2\omega_{(\bar{P},S(Q))} \hat{\times} {}^2\omega_{(\bar{P},S(Q))}) \quad A.3.10$$

and therefore we get $\gamma_{(P,S,Q)}(\varphi)^2 = 1$ which forces $\gamma_{(P,S,Q)}(\varphi) = 1$. We are done.

A.4. Since the unique *odd-polarization* ${}^2\omega$ in Theorem 4.3 can be easily extended to a *polarization*, it follows from Theorem 6.5 above that it supplies a *natural* bijection

$$\text{Irr}_k(\hat{G}) \cong \text{Wgt}_k(\hat{G}) \quad A.4.1$$

and we claim that this bijection coincides with the bijection defined by Gabriel Navarro in [7, Theorem 4.3] for $\pi = \{p\}$. First of all, borrowing all the notation in §7, suitably translated to our present situation, and choosing this *odd-polarization* ${}^2\omega$, we claim that the corresponding automorphism ${}^2\omega\theta$ of $\bar{N}_G(\mathcal{R})$ in 7.21 above is the identity map and therefore, according to Theorem 7.19 and Corollary 7.22, in this case we have

$$U \cong \text{Ind}_{\bar{N}_G(R_E)}^{\bar{N}_G(R)} ({}^2\omega V) \quad A.4.2.$$

That is to say, in the bijection A.4.1 determined by ${}^2\omega$ the image of any simple $k_*\hat{G}$ -module M can be directly computed from the triple formed by a *vertex* R , an *R-source* E and a *multiplicity* $\hat{N}_G(R_E)$ -module V of M .

A.5. More precisely, for any $n \in \mathbb{N}$, we claim that the automorphism ${}^2\omega\theta_n$ of $\hat{N}_G^n(\mathcal{R}) \star \hat{N}_G^{n+1}(\mathcal{R})^\circ$ is the identity map; indeed, since (cf. 7.13)

$$\text{Res}_{\rho_n}(T_n) \cong \text{Res}_{\rho_n}(S_n) \otimes_k \text{Res}_{\rho_{n+1}}(T_{n+1}) \quad A.5.1,$$

up to a suitable identification, from Theorem A.3 above we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{O}^2(\hat{F}_{S_n}(R^n)) \hat{\cap} \mathbb{O}^2(\hat{F}_{T_{n+1}}(R^n)) & \longrightarrow & \mathbb{O}^2(\hat{F}_{T_n}(R^n)) \\ {}^2\omega_{(R^n, S_n)} \hat{\times} {}^2\omega_{(R^n, T_{n+1})} \searrow & & \swarrow {}^2\omega_{(R^n, T_n)} \\ & k^* & \end{array} \quad A.5.2$$

which, according to the very definition of ${}^2\omega\theta_n$, proves our claim.

A.6. In particular, if \hat{G}' is a k^* -subgroup of \hat{G} and M' a $k_*\hat{G}'$ -module such that $M \cong \text{Ind}_{\hat{G}'}^{\hat{G}}(M')$, then M' is clearly a simple $k_*\hat{G}'$ -module and it is easily checked that a vertex R' and an R' -source E' of M' are also a vertex and an R' -source of M ; moreover, we claim that *if V' is a multiplicity $\hat{N}_{G'}(R'_{E'})$ -module of M' then the $k_*\hat{N}_G(R'_{E'})$ -module*

$$V = \text{Ind}_{\hat{N}_{G'}(R'_{E'})}^{\hat{N}_G(R'_{E'})}(V') \quad \text{A.6.1}$$

is a multiplicity $\hat{N}_G(R'_{E'})$ -module of M . Indeed, recall that we have (cf. 2.12)

$$\text{Ind}_{\hat{G}'}^{\hat{G}}(\text{End}_k(M')) \cong \text{End}_k(M) \quad \text{A.6.2}$$

and that id_M is the image of $\text{Tr}_{\hat{G}'}^{\hat{G}}(1 \otimes \text{id}_{M'} \otimes 1)$, and denote by $G'_{M'}$ the pointed group on $\text{End}_k(M)$ determined by the group G' and the idempotent $1 \otimes \text{id}_{M'} \otimes 1$. Since $R'_{E'}$ is a local pointed group on $\text{End}_k(M)$, the unity element in $(\text{End}_k(M))(R'_{E'})$ coincides with the sum $\sum_x x \otimes \text{id}_{M'} \otimes x^{-1}$ where x runs over the elements fulfilling $(R'_{E'})^x \subset G'_{M'}$ in a set of representatives for \hat{G}/\hat{G}' in \hat{G} and, for such an element x , $x \otimes \text{id}_{M'} \otimes x^{-1}$ denotes the image of $x \otimes \text{id}_{M'} \otimes x^{-1}$ in $(\text{End}_k(M))(R'_{E'})$ [9, Proposition 1.3]. But, it is clear that $(R'_{E'})^x$ is also a maximal local pointed group on $\text{End}_k(M')$ and therefore there is $x' \in \hat{G}'$ such that $(R'_{E'})^x = (R'_{E'})^{x'}$. Consequently, it follows from [6, statement 2.13.2] that we get an $\hat{N}_G(R'_{E'})$ -interior algebra isomorphism

$$\text{Ind}_{\hat{N}_{G'}(R'_{E'})}^{\hat{N}_G(R'_{E'})}((\text{End}_k(M'))(R'_{E'})) \cong (\text{End}_k(M))(R'_{E'}) \quad \text{A.6.3}$$

which proves our claim.

A.7. Moreover, by the very definitions of ${}^2\omega V$ and ${}^2\omega V'$ in 7.20 above, then we still have

$${}^2\omega V = \text{Ind}_{\bar{N}_{G'}(R'_{E'})}^{\bar{N}_G(R'_{E'})}({}^2\omega V') \quad \text{A.7.1}$$

and therefore it follows from isomorphism A.4.2 that we have a $k_*\bar{N}_{\hat{G}}(R')$ -module isomorphism

$$U \cong \text{Ind}_{\bar{N}_{G'}(R')}^{\bar{N}_{\hat{G}}(R')}(U') \quad \text{A.7.2}$$

where U' is a *simple projective* $\bar{N}_{G'}(R')$ -module which, together with R' , determines the G' -conjugacy class of *weights* of \hat{G}' determined by M' via the corresponding bijection A.4.1.

A.8. In conclusion, in order to prove that Navarro's correspondence in [7, Theorem 4.3] also maps M on the G -conjugacy class of the *weight* of \hat{G} determined by R and U , we may assume that M is *primitive* — namely,

that it is *not* induced from any proper k^* -subgroup of \hat{G} . In this case, as we mention in 1.4 above, it follows from [17, Lemma 30.4] that there is a G -stable finite p' -subgroup K of $\text{End}_k(M)^*$ which generates the k -algebra $\text{End}_k(M)$; in particular, $\text{End}_k(M)$ is actually a *Dade R -algebra* [13, 1.3], R is a Sylow p -subgroup of G and $\bar{N}_G(R)$ -stabilizes the isomorphism class of E [13, 1.8], so that $U \cong {}^{2\omega}V$ (cf. A.4.2). At this point, a careful inspection of the origin of Navarro's correspondence in [7, Theorem 3.1] shows that it maps M on the G -conjugacy class of the *weight* of \hat{G} determined by R and U if, for a suitable Brauer character ψ over $N_{\hat{G}}(R)$, we have

$$\text{Res}_{N_{\hat{G}}(R)}^{\hat{G}}(\varphi_M) = \varphi_U + 2 \cdot \psi \quad \text{A.8.1}$$

where φ_M and φ_U respectively denote the Brauer characters of M and of the $k_*N_{\hat{G}}(R)$ -module $U \cong {}^{2\omega}V$. Then, the fact that in our situation such an equality holds is more or less a consequence of [5, Theorem 5.3] but here we give a direct proof.

Proposition A.9. *Let M be a simple primitive $k_*\hat{G}$ -module, R a vertex, E an R -source and V a multiplicity $k_*\hat{N}_G(R_E)$ -module of M . Consider the unique odd-polarization ${}^{2\omega}$ such that diagram 4.3.1 is commutative and denote by ${}^{2\omega}V$ the restriction of V throughout the isomorphism $\bar{N}_{\hat{G}}(R) \cong \hat{N}_G(R_E)$ determined by ${}^{2\omega}$, and by φ_M and $\varphi_{{}^{2\omega}V}$ the respective Brauer characters of M and of ${}^{2\omega}V$ considered as a $k_*N_{\hat{G}}(R)$ -module. Then, for a suitable Brauer character ψ over $N_{\hat{G}}(R)$, we have*

$$\text{Res}_{N_{\hat{G}}(R)}^{\hat{G}}(\varphi_M) = \varphi_{{}^{2\omega}V} + 2 \cdot \psi \quad \text{A.9.1.}$$

Proof: Arguing by induction on $|G|$, we may assume that M is a faithful $k_*\hat{G}$ -module, then identifying \hat{G} with its image in $\text{End}_k(M)$; moreover, the case where $\dim_k(M) = 1$ being clear, we assume that $\dim_k(M) \neq 1$. Then, a minimal normal nontrivial subgroup K of G is an Abelian ℓ -elementary group for a prime number $\ell \neq p$ and the *primitivity* of M forces the converse image \hat{K} of K in \hat{G} to be the *central* product of k^* by an *extra-special* normal ℓ -subgroup of \hat{G} [4, Ch. 5, §5].

Let S be the k -subalgebra of $\text{End}_k(M)$ generated by \hat{K} ; once again, the *primitivity* of M forces S to be a simple k -algebra and then the k^* -quotient G of \hat{G} acts on S determining a k^* -group \hat{G} together with a k^* -group homomorphism $\hat{G} \rightarrow S^*$ (cf. 2.3), and we set

$$\hat{G} = \hat{G} \star (\hat{G})^\circ \quad \text{A.9.2;}$$

then, it follows from [16, Proposition 3.2] that there exists a $k_*\hat{G}$ -module \bar{M} such that we have a \hat{G} -interior algebra isomorphism

$$\text{End}_k(M) \cong S \otimes_k \text{End}_k(\bar{M}) \quad A.9.3;$$

actually, \hat{K} is *canonically* isomorphic to the converse image of K in \hat{G} and therefore K lifts to a normal subgroup of \hat{G} acting trivially on \bar{M} ; thus, up to suitable identifications, setting $\hat{\bar{G}} = \hat{G}/K$ and $S = \text{End}_k(N)$, \bar{M} becomes a $k_*\hat{\bar{G}}$ -module, we have a $k_*\hat{\bar{G}}$ -module isomorphism

$$M \cong N \otimes_k \bar{M} \quad A.9.4$$

and, denoting by φ_N the Brauer character of N and by $\hat{\pi}: \hat{G} \rightarrow \hat{\bar{G}}$ the canonical k^* -group homomorphism, we have

$$\varphi_M = \varphi_N \cdot \text{Res}_{\hat{\pi}}(\varphi_{\bar{M}}) \quad A.9.5.$$

Now, it is clear that \bar{M} is a simple primitive $k_*\hat{\bar{G}}$ -module, that the image \bar{R} of R in $\bar{G} = G/K$ is a vertex of \bar{M} (actually, it is a Sylow p -subgroup of \bar{G}), that we have a canonical R -interior algebra *embedding* (cf. 2.4)

$$\text{End}_k(E) \longrightarrow S \otimes_k \text{End}_k(\bar{E}) \quad A.9.6$$

where \bar{E} denotes an \bar{R} -source of \bar{M} , and that we still have a $\hat{N}_G(R_E)$ -interior algebra isomorphism [12, Proposition 5.6]

$$(\text{End}_k(M))(R_E) \cong S(R) \otimes_k (\text{End}_k(\bar{M}))(\bar{R}_{\bar{E}}) \quad A.9.7,$$

together with a k^* -group isomorphism [12, Proposition 5.11]

$$\hat{N}_G(R_E) \cong \hat{N}_G^S(R) \star \text{Res}_{\pi}(\hat{N}_{\bar{G}}(\bar{R}_{\bar{E}})) \quad A.9.8$$

where $\hat{N}_G^S(R)$ and $\text{Res}_{\pi}(\hat{N}_{\bar{G}}(\bar{R}_{\bar{E}}))$ respectively denote the k^* -groups coming from the action of $\bar{N}_G(R)$ on the simple k -algebra $S(R)$ [13, 1.8], and obtained by *pull-back* from the canonical group homomorphism $\pi: \bar{N}_G(R) \rightarrow \bar{N}_{\bar{G}}(\bar{R})$.

On the one hand, denoting by \bar{V} a *multiplicity* $\hat{N}_{\bar{G}}(\bar{R}_{\bar{E}})$ -module of \bar{M} , so that we have

$$\text{End}_k(\bar{V}) \cong (\text{End}_k(\bar{M}))(\bar{R}_{\bar{E}}) \quad A.9.9,$$

it follows from the induction hypothesis that, for a suitable Brauer character $\bar{\psi}$ over $N_{\hat{\bar{G}}}(\bar{R})$, we have

$$\text{Res}_{N_{\hat{\bar{G}}}(\bar{R})}^{\hat{\bar{G}}}(\varphi_{\bar{M}}) = \varphi_{2_{\omega}\bar{V}} + 2 \cdot \bar{\psi} \quad A.9.10.$$

On the other hand, denoting by W a *multiplicity* $\hat{N}_G^S(R)$ -module of N , it follows from isomorphisms A.9.7 and A.9.8 that we have a $k_*\hat{N}_G(R_E)$ -module isomorphism

$$V \cong W \otimes_k \text{Res}_\pi(\bar{V}) \quad \text{A.9.11;}$$

moreover, it follows from the commutativity of diagram A.3.1 that we still have a $k_*\bar{N}_G(R)$ -module isomorphism

$${}^2\omega V \cong {}^2\omega W \otimes_k \text{Res}_{\hat{\pi}_R}({}^2\omega \bar{V}) \quad \text{A.9.12}$$

where $\hat{\pi}_R$ denotes the restriction to $\bar{N}_G(R_E)$ of $\hat{\pi}$; consequently, with evident notation, we get

$$\varphi {}^2\omega V = \varphi {}^2\omega W \cdot \text{Res}_{\hat{\pi}_R}(\varphi {}^2\omega \bar{V}) \quad \text{A.9.13.}$$

But, according to Theorem A.10 below, we also have

$$\text{Res}_{\hat{N}_G(R)}^{\hat{G}}(\varphi_N) = \varphi {}^2\omega W + 2 \cdot \eta \quad \text{A.9.14}$$

for a suitable Brauer character η over $N_G(R)$. In conclusion, from equalities A.9.5, A.9.10 and A.9.14 we get

$$\begin{aligned} \text{Res}_{\hat{N}_G(R)}^{\hat{G}}(\varphi_M) &= \text{Res}_{\hat{N}_G(R)}^{\hat{G}}(\varphi_N) \cdot \text{Res}_{\hat{\pi}_R}(\text{Res}_{\hat{N}_G(R)}^{\hat{G}}(\varphi_{\bar{M}})) \\ &= (\varphi {}^2\omega W + 2 \cdot \eta) \cdot \text{Res}_{\hat{\pi}_R}(\varphi {}^2\omega \bar{V} + 2 \cdot \bar{\psi}) \\ &= \varphi {}^2\omega V + 2 \cdot \psi \end{aligned} \quad \text{A.9.15}$$

where $\psi = \eta \cdot \text{Res}_{\hat{\pi}_R}(\varphi {}^2\omega \bar{V} + 2 \cdot \bar{\psi}) + \varphi {}^2\omega W \cdot \text{Res}_{\hat{\pi}_R}(\bar{\psi})$. We are done.

Theorem A.10. *Let M be a $k_*\hat{G}$ -module such that $\text{End}_k(M)$ is generated by a G -stable k^* -subgroup \hat{K} of $\text{End}_k(M)^*$ which is the central product of k^* by an extra-special ℓ -subgroup for an odd prime number $\ell \neq p$. For any local pointed group R_E on $\text{End}_k(M)$, denoting by V a multiplicity $\hat{N}_G(R_E)$ -module of R_E and by ${}^2\omega V$ the restriction of V via the isomorphism $\bar{N}_G(R) \cong \hat{N}_G(R_E)$ determined by the unique odd-polarization ${}^2\omega$ such that diagram 4.3.1 is commutative, we have*

$$\text{Res}_{\hat{N}_G(R)}^{\hat{G}}(\varphi_M) = \varphi {}^2\omega V + 2 \cdot \psi \quad \text{A.10.1}$$

where φ_M and $\varphi {}^2\omega V$ denote the respective Brauer characters of M and of ${}^2\omega V$ considered as a $k_*\bar{N}_G(R)$ -module, and ψ is a Brauer character over $N_G(R)$.

Proof: We actually may assume that M is faithful and that $\hat{G} = \bar{N}_G(R)$; then, G stabilizes the decomposition [4, Ch. 5, Theorem 2.3]

$$K = C_K(R) \times [K, R] \quad \text{A.10.2}$$

of the k^* -quotient of \hat{K} ; thus, setting $S = \text{End}_k(M)$ and denoting by S' and S'' the k -subalgebras of S generated by the respective converse images \hat{K}' of $C_K(R)$ and \hat{K}'' of $[K, R]$, we have $S = S' \otimes_k S''$, \hat{K}' and \hat{K}'' are also central products of k^* by extra-special ℓ -subgroups (here we also consider $\mathbb{Z}/\ell\mathbb{Z}$ as an *extra-special* ℓ -group) and G still stabilizes

$$S' = \text{End}_k(M') \quad \text{and} \quad S'' = \text{End}_k(M'') \quad \text{A.10.3.}$$

Consequently, as in the proof above, it follows from the commutativity of diagram A.3.1 that it suffices to prove the theorem for M' and for M'' . That is to say, we may assume that either $K = C_K(R)$ or $K = [K, R]$; in the first case, R centralizes \hat{K} [4, Ch. 5, Theorem 1.4], so that it centralizes S which forces $R = \{1\}$; then, we have $M = V$, $E = k$ and $\hat{F}_S(R) = k^*$, and by the very definition of $\hat{N}_G(R_E)$ (cf. 2.5) we get an isomorphism $\hat{N}_G(R_E) \cong \bar{N}_{\hat{G}}(R) = \hat{G}$ compatible with the canonical k^* -group homomorphism 2.8.1, so that equality A.10.1 is trivially true with $\psi = 0$.

Following the notation in A.11 and according to isomorphism A.12.2 below, let us denote by H the image of $Sp(K, \kappa)$ in $N_{S^*}(\hat{K})$; in particular, the nontrivial element in $Z(H)$ is an involution $s \in S$ which stabilizes \hat{K} and induces $-\text{id}_K$ over K ; note that, if $s' \in S$ is such an involution then s' stabilizes \hat{K} acting trivially on K and therefore, according again to isomorphism A.12.2 below, s' belongs to $\hat{K} \cdot \langle s \rangle$, so that we have $s' \in \{s^x, -s^x\}$ for a suitable $x \in \hat{K}$.

In the second case above, we have $C_K(R) = \{1\}$ and therefore R fixes a unique pair of such involutions $\{s, -s\}$ which by oddness forces

$$\hat{G} = N_{\hat{G}}(R) \subset C_{S^*}(s) \quad \text{A.10.4;}$$

consequently, \hat{G} is contained in the intersection

$$N_{S^*}(\hat{K}) \cap C_{S^*}(s) = k^* \times H \quad \text{A.10.5.}$$

Moreover, since K indexes an R -stable basis of $S = k_* \hat{K}$ (cf. A.11 below), we have $S(R) \cong k$ which forces $V \cong k$; hence, by the very definition of the k^* -group $\hat{N}_G(R_E)$, in this case we get a k^* -group isomorphism (cf. 2.5)

$$\hat{N}_G(R_E) \cong k^* \times \bar{N}_G(R) \quad \text{A.10.6.}$$

At this point, it follows from Lemma A.14 below that the decomposition

$$\hat{G} = N_{\hat{G}}(R) \cong \hat{N}_G(R_E) \cong k^* \times G \quad \text{A.10.7}$$

determined by the k^* -group homomorphism ${}^2\omega_{(R,S)}: \hat{F}_S(R) \rightarrow k^*$ coincides

with the decomposition

$$\hat{G} = k^* \times (\hat{G} \cap H) \quad A.10.8$$

obtained from the inclusion $\hat{G} \subset N_{S^*}(\hat{K}) \cap C_{S^*}(s)$. In particular, the restriction of $\varphi_{2\omega_V}$ to $\hat{G} \cap H$ is just the *trivial* character.

On the other hand, for any $y \in \hat{G} \cap H$, acting over K the product sy only fix the trivial element 1; indeed, if $syx(sy)^{-1} = x$ for some $x \in K$ then $yx^{-1}y^{-1} = x$ and therefore $\{x, x^{-1}\}$ is an orbit of $\langle y \rangle$ which forces $x = x^{-1}$, so that $x = 1$; in particular, we get

$$\text{tr}_M(sy) \cdot \text{tr}_{M^*}(sy) = \text{tr}_S(sy) = 1 \quad A.10.9.$$

But, we clearly have

$$k\langle s \rangle = k \cdot \text{id}_M + k \cdot s = k \cdot i + k \cdot i' \quad A.10.10$$

for suitable mutually orthogonal idempotents i and i' of S , and we choose the notation in such a way that $\dim(i(M)) \geq \dim(i'(M))$.

Then, denoting by φ_M , $\varphi_{i(M)}$ and $\varphi_{i'(M')}$ the respective Brauer characters of M , $i(M)$ and $i'(M')$, let us consider the ordinary characters χ_M , $\chi_{i(M)}$ and $\chi_{i'(M')}$ over $\hat{G} \cap H$ which respectively lift the restrictions to $\hat{G} \cap H$ of φ_M , $\varphi_{i(M)}$ and $\varphi_{i'(M')}$ to the set of characters χ fulfilling $\chi(y) = \chi(y_{p'})$ for any $y \in \hat{G} \cap H$; consequently, we clearly have $\chi_M = \chi_{i(M)} + \chi_{i'(M')}$ and moreover

$$1 = (\chi_{i(M)}(y) - \chi_{i'(M)}(y))(\bar{\chi}_{i(M)}(y) - \bar{\chi}_{i'(M)}(y)) \quad A.10.11$$

for any $y \in \hat{G} \cap H$ (cf. A.10.9); in particular, the *norm* of $\chi_{i(M)} - \chi_{i'(M)}$ is equal to 1 and, according to our choice of notation, we still have

$$1 = \chi_{i(M)}(1) - \chi_{i'(M)}(1) \quad A.10.12$$

hence, for a suitable linear character λ of $\hat{G} \cap H$, we get $\chi_{i(M)} = \lambda + \chi_{i'(M)}$ or, equivalently,

$$\chi_M = \lambda + 2 \cdot \chi_{i'(M)} \quad A.10.13.$$

Now, it suffices to prove that λ is the trivial character. Note that, denoting by k' the subfield of k generated by the ℓ -th roots of unity, we still can define a k'^* -group $\hat{K}' = k'^* \times K$ as in A.11.1 below and, setting $S' = k'_* \hat{K}'$, we have $S = k \otimes_{k'} S'$ and H is contained in $1 \otimes S'$, so that i and i' also belong to $1 \otimes S'$; hence, the values of the ordinary characters χ_M , $\chi_{i(M)}$ and $\chi_{i'(M')}$ are contained in the extension of \mathbb{Q} by the ℓ -th roots of unity. Consequently, it suffices to prove that the restriction of λ to a Sylow ℓ -subgroup L of $\hat{G} \cap H$ is trivial.

But, it is well-known that for a maximal Abelian k^* -subgroup \hat{A} of \hat{K} and a k^* -group homomorphism $\zeta : \hat{A} \rightarrow k^*$, denoting by k_ζ the corresponding $k_*\hat{A}$ -module, we have

$$M \cong \text{Ind}_{\hat{A}}^{\hat{K}}(k_\zeta) \quad \text{A.10.14;}$$

moreover, since H acts over $\hat{K} \cong k^* \times K$ stabilizing $1 \times K$, it is easily checked that L stabilizes a suitable choice of $\text{Ker}(\zeta) \subset 1 \times K$ and therefore, choosing a complement X of $\text{Ker}(\zeta)$ in K , it stabilizes the basis $\{(1, x) \otimes 1\}_{x \in X}$ of M ; then, L fixes $(1, 1) \times 1$ and, for any L -orbit O in $X - \{1\}$, $\{(1, x) \otimes 1\}_{x \in O}$ and $\{(1, x^{-1}) \otimes 1\}_{x \in O}$ are *different* orbits of L in this basis, since $|O|$ is odd; that is to say, the number of orbits of L in this basis is odd.

In conclusion, since L is an ℓ -group and $\ell \neq p$, the multiplicity of k in M considered as a kL -module is an odd number; then, the restriction of equality A.10.13 to L proves that the restriction of λ to L is trivial. We are done.

A.11. Let ℓ be an odd prime number different from p and \hat{K} a k^* -group which is the central product of k^* by an extra-special ℓ -group and, for our purposes, we also consider $\mathbb{Z}/\ell\mathbb{Z}$ as an *extra-special* ℓ -group. Denote by κ the *non-singular skew symmetric scalar product* over the k^* -quotient K induced by the commutator in \hat{K} ; thus, we have $|K| = \ell^{2n}$ and note that the case $n = 0$ is not excluded. Then, it is easily checked that \hat{K} is isomorphic to $k^* \times K$ endowed with the product defined by

$$(\lambda, x) \cdot (\lambda', x') = (\lambda\lambda'\kappa(x, x')^{\frac{1}{2}}, xx') \quad \text{A.11.1,}$$

for any $\lambda, \lambda' \in k^*$ and any $x, x' \in K$, and with the group homomorphism $k^* \rightarrow k^* \times K$ mapping $\lambda \in k^*$ on $(\lambda, 1)$. It is quite clear that the corresponding symplectic group $Sp(K, \kappa)$ acts over this k^* -group and we actually have

$$\text{Aut}_{k^*}(\hat{K}) \cong K \rtimes Sp(K, \kappa) \quad \text{A.11.2.}$$

A.12. Moreover, it is well-known that $S = k_*\hat{K}$ is a simple k -algebra and $Sp(K, \kappa)$ clearly acts over this k -algebra stabilizing \hat{K} ; thus, since any central k^* -extension of $Sp(K, \kappa)$ is trivial, this action can be lifted to a group homomorphism

$$Sp(K, \kappa) \longrightarrow N_{S^*}(\hat{K}) \quad \text{A.12.1;}$$

then, since $C_{S^*}(\hat{K}) = k^* \cdot \text{id}_S$, from isomorphism A.11.2 we easily get

$$N_{S^*}(\hat{K}) \cong \hat{K} \rtimes Sp(K, \kappa) \quad \text{A.12.2;}$$

let us identify $Sp(K, \kappa)$ with its image in $N_{S^*}(\hat{K})$ (for the choice of a k^* -group isomorphism $\hat{K} \cong k^* \times K$!).

A.13. Identifying \hat{K} with $k^* \times K$, it is clear that a p -subgroup R of $Sp(K, \kappa)$ stabilizes the basis $\{(1, x)\}_{x \in K}$ of S and therefore S becomes a *Dade R -algebra*; moreover, it is easily checked that the restriction κ_R of κ to $C_K(R)$ remains a *non-singular skew symmetric scalar product* and therefore $C_{\hat{K}}(R) \cong k^* \times C_K(R)$ is also the central product of k^* by an extra-special ℓ -group. Then, it is clear that the Brauer homomorphism induces a k -algebra isomorphism [11, statement 2.8.4]

$$k_* C_{\hat{K}}(R) \cong S(R) \quad A.13.1$$

and it is easily checked that the action of $N_{Sp(K, \kappa)}(R)$ over $C_{\hat{K}}(R)$ is contained in the corresponding symplectic group $Sp(C_K(R), \kappa_R)$; that is to say, the Brauer homomorphism can be extended to a group homomorphism

$$\text{Br}_R^* : N_{Sp(K, \kappa)}(R) \longrightarrow Sp(C_K(R), \kappa_R) \subset S(R)^* \quad A.13.2$$

such that the action of $x \in N_{Sp(K, \kappa)}(R)$ coincides with the conjugation by $\text{Br}_R^*(x)$ on $S(R)$; thus, choosing an element $a \in S^R$ lifting $\text{Br}_R^*(x)$ and an idempotent i in the unique local point of R on S , and denoting by \tilde{x}^R the image of x in $F_S(R)$ and by $\overline{ixa^{-1}i}^S$ the image of the product $ixa^{-1}i$ in the quotient

$$N_{(iSi)^*}(Ri) / \left(i + J((iSi)^R) \right) \quad A.13.3,$$

the pair $(\tilde{x}^R, \overline{ixa^{-1}i}^S)$ is an element of $\hat{F}_S(R)$ [11, Proposition 6.10].

Lemma A.14. *With the notation above, denote by ${}^2\omega$ the unique odd-polarization such that diagram 4.3.1 is commutative and let R be a p -subgroup of $Sp(K, \kappa)$. For any $x \in N_{Sp(K, \kappa)}(R)$ of odd order, choosing an element $a \in S^R$ lifting $\text{Br}_R^*(x)$ and an idempotent i in the unique local point of R on S , and denoting by \tilde{x}^R the image of x in $F_S(R)$ and by $\overline{ixa^{-1}i}^S$ the image of the product $ixa^{-1}i$ in the quotient*

$$N_{(iSi)^*}(Ri) / \left(i + J((iSi)^R) \right) \quad A.14.1,$$

we have

$${}^2\omega_{(R, S)}(\tilde{x}^R, \overline{ixa^{-1}i}^S) = 1 \quad A.14.2.$$

Proof: Arguing by induction on $|R|$, set $Z = \Omega_1(Z(R))$, $\bar{R} = R/Z$ and $T = \text{Res}_Z^R(S)$; it follows from Theorem A.3 above that we have the commutative diagram

$$\begin{array}{ccc} \mathbb{O}^2(\hat{F}_S(R)_Z) & \xrightarrow{\Delta_{R, S, Z}} & \mathbb{O}^2(\hat{F}_T(Z)) \hat{\times} \mathbb{O}^2(\hat{F}_S(Z)(\bar{R})) \\ {}^2\omega_{(R, S)} \searrow & & \swarrow {}^2\omega_{(Z, T)} \hat{\times} {}^2\omega_{(R, S(Z))} \\ & k^* & \end{array} \quad A.14.3.$$

But, choosing an element $c \in S^Z$ lifting $\bar{x} = \text{Br}_Z^*(x)$ and an idempotent j in the unique local point of Z on S fulfilling $ji = j = ij$, and setting $\bar{a} = \text{Br}_Z(a)$ and $\bar{i} = \text{Br}_Z(i)$, it is easily checked from [15, Proposition 9.11] that we have

$$\Delta_{R,S,Z}(\tilde{x}^R, \overline{ixa^{-1}i}^S) = (\tilde{x}^Z, \overline{jxc^{-1}j}^T) \cdot (\tilde{x}^{\bar{R}}, \overline{i\bar{x}\bar{a}^{-1}\bar{i}}^{S(Z)}) \quad \text{A.14.4;}$$

hence, since we clearly have

$$\text{Br}_{\bar{R}}^*(\text{Br}_Z^*(x)) = \text{Br}_R^*(x) \quad \text{A.14.5,}$$

if $Z \neq R$ then from the induction hypothesis we get

$${}^2\omega_{(Z,T)}(\tilde{x}^Z, \overline{jxc^{-1}j}^T) = 1 = {}^2\omega_{(\bar{R},S(Z))}(\tilde{x}^{\bar{R}}, \overline{i\bar{x}\bar{a}^{-1}\bar{i}}^{S(Z)}) \quad \text{A.14.6.}$$

Now, equality A.14.2 follows from the commutativity of diagram A.14.3.

From now on, we assume that R is p -elementary Abelian. Arguing by induction on $|K|$, if K decomposes on a *direct orthogonal* sum of two $R \cdot \langle x \rangle$ -stable nontrivial subspaces

$$K = K' \perp K'' \quad \text{A.14.7}$$

then \hat{K} is the central product of the converse images \hat{K}' of K' and \hat{K}'' of K'' , and, setting

$$S' = k_* \hat{K}' \quad \text{and} \quad S'' = k_* \hat{K}'' \quad \text{A.14.8,}$$

S' and S'' are also *Dade R -algebras* and we have $S \cong S' \otimes_k S''$; in particular, it follows from Theorem A.3 above that we have the commutative diagram

$$\begin{array}{ccc} \mathbb{O}^2(\hat{F}_{S'}(R)) \hat{\cap} \mathbb{O}^2(\hat{F}_{S''}(R)) & \xrightarrow{\hat{\nu}_{R,S',S''}} & \mathbb{O}^2(\hat{F}_S(R)) \\ {}^2\omega_{(R,S')} \hat{\times} {}^2\omega_{(R,S'')} & \searrow & \swarrow {}^2\omega_{(R,S)} \\ & k^* & \end{array} \quad \text{A.14.9.}$$

But, denoting by κ' and κ'' the respective restrictons of κ to K' and K'' , it is clear that x is the image of $x' \otimes x''$ for suitable elements $x' \in Sp(K', \kappa')$ and $x'' \in Sp(K'', \kappa'')$ normalizing the respective images $R' \subset Sp(K', \kappa')$ and $R'' \subset Sp(K'', \kappa'')$ of R . Moreover, choosing elements $a' \in S'^{R'}$ and $a'' \in S''^{R''}$ respectively lifting $\text{Br}_{R'}^*(x')$ and $\text{Br}_{R''}^*(x'')$, and idempotents i' and i'' in the respective unique local points of R on S' and on S'' , we clearly may choose the element a equal to the image of $a' \otimes a''$ and the idempotent i in an orthogonal decomposition of the image of $i' \otimes i''$, so that $\text{Br}_R(x)$ is equal to the corresponding image of $\text{Br}_{R'}(x') \otimes \text{Br}_{R''}(x'')$ *via* the isomorphism

$$S'(R') \otimes_k S''(R'') \cong S(R) \quad \text{A.14.10.}$$

Then, it easily follows from [15, 9.15] that we have

$$\begin{aligned} \hat{\nu}_{R,S',S''}((\widetilde{x'}^{R'}, \overline{i'x'a'^{-1}i'}^{S'}) \cdot (\widetilde{x''}^{R''}, \overline{i''x''a''^{-1}i''}^{S''})) \\ = (\widetilde{x}^R, \overline{ixa^{-1}i}^S) \end{aligned} \quad A.14.11.$$

Now, since the induction hypothesis implies that

$${}^2\omega_{(R',S')}(\widetilde{x'}^{R'}, \overline{i'x'a'^{-1}i'}^{S'}) = 1 = {}^2\omega_{(R'',S'')}(\widetilde{x''}^{R''}, \overline{i''x''a''^{-1}i''}^{S''}) \quad A.14.12,$$

equality A.14.2 follows from the commutativity of diagram A.14.9.

Thus, we may assume that K does not admit a decomposition on a *direct orthogonal* sum of two $R \cdot \langle x \rangle$ -stable nontrivial subspaces. Denoting by \mathbb{F} the field of cardinal ℓ , if L is a simple $\mathbb{F}R$ -submodule of K then the restriction of κ to L is either *non-singular* or, denoting by L^\perp the *orthogonal* space of L , L^\perp contains L and we have a canonical isomorphism $K/L^\perp \cong L^*$, so that, if L' is an $\mathbb{F}R$ -complement of L^\perp in K , the restriction of κ to $L \oplus L'$ is *non-singular*. Consequently, the dimensions of all the simple $\mathbb{F}R$ -submodules of K have the same *parity*.

Firstly assume that the dimensions of all the simple $\mathbb{F}R$ -submodules L of K are *odd*; in this case, L^* is also a $\mathbb{F}R$ -submodule of K *not* isomorphic to L . Thus, since $|\langle x \rangle|$ is odd, the group $\langle x \rangle$ has exactly two orbits in the set of *isotypic components* of the $\mathbb{F}R$ -module K and then, denoting by A and B the sums of *isotypic components* in each $\langle x \rangle$ -orbit, A and B are maximal *totally singular* subspaces fulfilling $K = A \oplus B$. Hence, the converse images \hat{A} of A and \hat{B} of B in \hat{K} are maximal Abelian subgroups and it is well-known that, for a k^* -group homomorphism $\zeta : \hat{A} \rightarrow k^*$ that we may choose $R \cdot \langle x \rangle$ -stable (cf. A.11.1), we have

$$M \cong \text{Ind}_{\hat{A}}^{\hat{K}}(k_\zeta) \quad A.14.13$$

where k_ζ denotes the corresponding $k_*\hat{A}$ -module.

In this situation, the group $R \cdot \langle x \rangle$ stabilizes the basis $\{(1, y) \otimes 1\}_{y \in B}$ of M , so that the *Dade R -algebra* S is *similar* to k ; in particular, identifying S with the *induced \hat{K} -interior algebra* $\text{Ind}_{\hat{A}}^{\hat{K}}(k_\zeta)$ (cf. 2.12) where k_ζ still denotes the corresponding \hat{A} -interior algebra, the primitive idempotent

$$i = (1, 1) \otimes 1 \otimes (1, 1) \quad A.14.14$$

actually belongs to the unique local point of R on S ; now, x and $\text{Br}_R^*(x)$ respectively centralize i and $\text{Br}_R(i)$, and, with the notation above, it is easily checked that $\overline{ixa^{-1}i}^S = \bar{i}^S$ which proves equality A.14.2 in this case.

Finally assume that the dimensions of all the simple $\mathbb{F}R$ -submodules L of K are *even*; in this case, the image of $\mathbb{F}R$ in $\text{End}_{\mathbb{F}}(L)$ is an extension \mathbb{F}_L of \mathbb{F} of even degree and therefore it contains a *primitive fourth* root τ_L of unity;

moreover, since $|\langle x \rangle|$ is odd, the stabilizer in $\langle x \rangle$ of the *isotypic component* containing L acts on \mathbb{F}_L fixing τ_L . Consequently, considering all the orbits of $\langle x \rangle$, we get a *self-adjoint* endomorphism τ of K which centralizes $R \cdot \langle x \rangle$ and fulfills $\tau^2 = -\text{id}_K$.

At this point, we consider the central product $\hat{K} \hat{\times} \hat{K}$, and in the k^* -quotient $K \times K$ we set

$$A = \{(y, \tau(y))\}_{y \in K} \quad \text{and} \quad B = \{(-y, \tau(y))\}_{y \in K} \quad A.14.15;$$

as above, we have $K = A \oplus B$, A is *totally singular* since

$$\begin{aligned} (\kappa \times \kappa)((y, \tau(y)), (y', \tau(y'))) &= \kappa(y, y') \kappa(\tau(y), \tau(y')) \\ &= \kappa(y, y') \kappa(\tau^2(y), y') \\ &= \kappa(y, y') \kappa(y, y')^{-1} = 1 \end{aligned} \quad A.14.16$$

for any $y \in K$ and, similarly, B is *totally singular* too. Once again, the converse images \hat{A} of A and \hat{B} of B in $\hat{K} \hat{\times} \hat{K}$ are maximal Abelian subgroups; hence, the argument above applied to the p -subgroup $\Delta(R) = \{u \otimes u\}_{u \in R}$ and to the element $x \otimes x$ of $Sp(K \times K, \kappa \times \kappa)$ proves that

$${}^2\omega_{(\Delta(R), S \otimes_k S)}(\widetilde{x \otimes x}^{\Delta(R)}, \overline{j(x \otimes x)(a \otimes a)^{-1}j}^{S \otimes_k S}) = 1 \quad A.14.17$$

for the choice of an idempotent j in the unique local point of $\Delta(R)$ on $S \otimes_k S$.

Consequently, since it follows again from Theorem A.3 that we have the commutative diagram

$$\begin{array}{ccc} \mathbb{O}^2(\hat{F}_S(R)) \star \mathbb{O}^2(\hat{F}_S(R)) & \xrightarrow{\hat{\nu}_{R,S,S}} & \mathbb{O}^2(\hat{F}_{S \otimes_k S}(R)) \\ {}^2\omega_{(R,S)} \hat{\times} {}^2\omega_{(R,S)} \searrow & & \swarrow {}^2\omega_{(R,S \otimes_k S)} \\ & k^* & \end{array} \quad A.14.18$$

and since we clearly have

$$\begin{aligned} \hat{\nu}_{R,S,S}((\tilde{x}^R, \overline{ixa^{-1}i}^S), (\tilde{x}^R, \overline{ixa^{-1}i}^S)) \\ = (\widetilde{x \otimes x}^{\Delta(R)}, \overline{j(x \otimes x)(a \otimes a)^{-1}j}^{S \otimes_k S}) \end{aligned} \quad A.14.19,$$

from equality A.14.17 we actually get

$$({}^2\omega_{(R,S)}(\tilde{x}^R, \overline{ixa^{-1}i}^S))^2 = 1 \quad A.14.20$$

which forces ${}^2\omega_{(R,S)}(\tilde{x}^R, \overline{ixa^{-1}i}^S) = 1$ since x has odd order. We are done.

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